STEP III, 2004, Q7 MS

7 On $0 \leqslant t \leqslant 1$, the integrand is non-negative and $0 \leqslant \dfrac{t}{t+1} = 1 - \dfrac{1}{t+1} \leqslant \frac{1}{2}$, so

$$I_{n+1} = \int_0^1 \frac{t^n}{(t+1)^{n+1}}\,dt = \int_0^1 \frac{t}{t+1}\,\frac{t^{n-1}}{(t+1)^n}\,dt < \frac{1}{2}\int_0^1 \tfrac{1}{2}\frac{t^{n-1}}{(t+1)^n}\,dt = \tfrac{1}{2}I_n.$$

Integration by parts gives

$$I_{n+1} = \left[-\frac{t^n}{n(t+1)^n}\right]_0^1 + \int_0^1 \frac{nt^{n-1}}{n(t+1)^n}\,dt = -\frac{1}{n2^n} + I_n,$$

so $I_n > 2I_{n+1} = -\dfrac{1}{n2^{n-1}} + 2I_n \quad \Rightarrow \quad I_n < \dfrac{1}{n2^{n-1}} \quad (*)$.

Since $\dfrac{1}{2^r} = I_r - I_{r+1}$, $\displaystyle\sum_{r=1}^{n} \frac{1}{r2^r} = (I_1 - I_2) + (I_2 - I_3) + \ldots + (I_n - I_{n+1}) = I_1 - I_{n+1}$, and

$I_1 = \displaystyle\int_0^1 \frac{1}{t+1}\,dt = \ln 2$, so $\ln 2 = \displaystyle\sum_{r=1}^{n} \frac{1}{r2^r} + I_{n+1}$.

Hence $\ln 2 > \displaystyle\sum_{r=1}^{3} \frac{1}{r2^r} = \frac{2}{3}$ and, by inequality $(*)$, $\ln 2 = \displaystyle\sum_{r=1}^{2} \frac{1}{r2^r} + I_3 < \sum_{r=1}^{2} \frac{1}{r2^r} + \frac{1}{3\cdot 2^2} = \frac{17}{24}$.