

### STEP III, 2004, Q7 MS

7 On  $0 \leq t \leq 1$ , the integrand is non-negative and  $0 \leq \frac{t}{t+1} = 1 - \frac{1}{t+1} \leq \frac{1}{2}$ , so

$$I_{n+1} = \int_0^1 \frac{t^n}{(t+1)^{n+1}} dt = \int_0^1 \frac{t}{t+1} \frac{t^{n-1}}{(t+1)^n} dt < \frac{1}{2} \int_0^1 \frac{t^{n-1}}{(t+1)^n} dt = \frac{1}{2} I_n.$$

Integration by parts gives

$$I_{n+1} = \left[ -\frac{t^n}{n(t+1)^n} \right]_0^1 + \int_0^1 \frac{nt^{n-1}}{n(t+1)^n} dt = -\frac{1}{n2^n} + I_n,$$

$$\text{so } I_n > 2I_{n+1} = -\frac{1}{n2^{n-1}} + 2I_n \Rightarrow I_n < \frac{1}{n2^{n-1}} \quad (*).$$

Since  $\frac{1}{2^r} = I_r - I_{r+1}$ ,  $\sum_{r=1}^n \frac{1}{r2^r} = (I_1 - I_2) + (I_2 - I_3) + \dots + (I_n - I_{n+1}) = I_1 - I_{n+1}$ , and

$$I_1 = \int_0^1 \frac{1}{t+1} dt = \ln 2, \text{ so } \ln 2 = \sum_{r=1}^n \frac{1}{r2^r} + I_{n+1}.$$

$$\text{Hence } \ln 2 > \sum_{r=1}^3 \frac{1}{r2^r} = \frac{2}{3} \text{ and, by inequality } (*), \ln 2 = \sum_{r=1}^2 \frac{1}{r2^r} + I_3 < \sum_{r=1}^2 \frac{1}{r2^r} + \frac{1}{3 \cdot 2^2} = \frac{17}{24}.$$



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