STEP III, 2004, Q7

7 For $n = 1, 2, 3, \ldots$, let

$$I_n = \int_0^1 \frac{t^{n-1}}{(t+1)^n}\,\mathrm{d}t\,.$$

By considering the greatest value taken by $\dfrac{t}{t+1}$ for $0 \leqslant t \leqslant 1$ show that $I_{n+1} < \frac{1}{2} I_n$.

Show also that $I_{n+1} = -\dfrac{1}{n\,2^n} + I_n$.

Deduce that $I_n < \dfrac{1}{n\,2^{n-1}}$.

Prove that

$$\ln 2 = \sum_{r=1}^{n} \frac{1}{r\,2^r} + I_{n+1}$$

and hence show that $\frac{2}{3} < \ln 2 < \frac{17}{24}$.