STEP III, 2004, Q6 MS

6 $F_n - F_{n-1} = w_n^2 + w_{n-1}^2 - 4w_n w_{n-1} - w_{n-1}^2 - w_{n-2}^2 + 4w_{n-1}w_{n-2} = w_n^2 - w_{n-2}^2 - 4w_{n-1}(w_n - w_{n-2})$
$= (w_n - w_{n-2})(w_n + w_{n-2} - 4w_{n-1}.)$ $(+)$

(i) Let w_n be u_n; then $u_n + u_{n-2} - 4u_{n-1} = 0$, so $F_n - F_{n-1} = 0$ for $n \geqslant 2$, by $(+)$,
but $F_1 = u_1^2 + u_0^2 - 4u_1u_0 = -3$ so $F_n = -3$ for $n \geqslant 1$

(ii) In this part, let w_n be v_n.

(a) $v_1^2 + 1 = 4v_1 - 3 \Rightarrow (v_1 - 2)^2 = 0 \Rightarrow v_1 = 2$
$F_n = v_n^2 + v_{n-1}^2 - 4v_n v_{n-1} = -3$ for $n \geqslant 1$
$\Rightarrow v_n - v_{n-2} = 0$ or $v_n + v_{n-2} - 4v_{n-1} = 0$, for $n \geqslant 2$, by $(+)$.

(b) Since $1, 2, 1, 2, \ldots$ satisfies $v_n - v_{n-2} = 0$ for $n \geqslant 2$, F_n is constant, by $(+)$ and since $v_0 = 1, v_1 = 2$ that constant is -3, so the sequence satisfies $(*)$.

(c) The sequence $1, 2, 7, 2, \ldots$, with period 4, satisfies $v_n - v_{n-2} = 0$ for odd $n \geqslant 2$ and $v_n + v_{n-2} - 4v_{n-1} = 0$ for even $n \geqslant 2$, so F_n is constant, by $(+)$, and since $v_0 = 1, v_1 = 2$ that constant is -3, so the sequence satisfies $(*)$.