STEP III, 2004, Q6

6 Given a sequence $w_0, w_1, w_2, \ldots$, the sequence $F_1, F_2, \ldots$ is defined by

$$F_n = w_n^2 + w_{n-1}^2 - 4w_n w_{n-1}\,.$$

Show that $F_n - F_{n-1} = (w_n - w_{n-2})\,(w_n + w_{n-2} - 4w_{n-1})$ for $n \geqslant 2$.

(i) The sequence $u_0, u_1, u_2, \ldots$ has $u_0 = 1$, and $u_1 = 2$ and satisfies

$$u_n = 4u_{n-1} - u_{n-2} \quad (n \geqslant 2)\,.$$

Prove that $u_n^2 + u_{n-1}^2 = 4u_n u_{n-1} - 3$ for $n \geqslant 1$.

(ii) A sequence $v_0, v_1, v_2, \ldots$ has $v_0 = 1$ and satisfies

$$v_n^2 + v_{n-1}^2 = 4v_n v_{n-1} - 3 \quad (n \geqslant 1). \qquad (*)$$

(a) Find v_1 and prove that, for each $n \geqslant 2$, either $v_n = 4v_{n-1} - v_{n-2}$ or $v_n = v_{n-2}$.

(b) Show that the sequence, with period 2, defined by

$$v_n = \begin{cases} 1 & \text{for } n \text{ even} \\ 2 & \text{for } n \text{ odd} \end{cases}$$

satisfies $(*)$.

(c) Find a sequence v_n with period 4 which has $v_0 = 1$, and satisfies $(*)$.