STEP III, 2004, Q5 MS

5 If $\cos(x-\alpha) = \cos\beta$ then $x - \alpha = 2n\pi \pm \beta$ so $x = \alpha \pm \beta + 2n\pi$ so $\tan x = \tan(\alpha \pm \beta)$ however, for example, $x = \pi$, $\alpha = \beta = 0$ has $\tan x = \tan \pi = \tan 0 = \tan(\alpha + \beta)$ but $\cos(x-\alpha) = \cos\pi = -1 \neq 1 = \cos\beta$.

a Writing $\cos x - 7\sin x = R\cos(x-\alpha)$ requires $R = \sqrt{50} = 5\sqrt{2}$ and $\tan\alpha = -7$, so $\cos(x-\alpha) = \cos\beta$, where $\cos\beta = \dfrac{1}{\sqrt{2}}$, so we can take $\tan\beta = 1$.

Hence $\tan x = \tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{-7 \pm 1}{1 \pm 7} = -\frac{3}{4}$ or $\frac{4}{3}$.

The first of these gives $x = \frac{1}{2}\pi + \omega$ or $x = \frac{3}{2}\pi + \omega$ (since $\arctan\frac{3}{4} = \frac{\pi}{2} - \arctan\frac{4}{3}$)and the second $x = \omega$ or $x = \pi + \omega$. However, the first solution in each case does not satisfy the original equation (both have $\sin x > 0$, so $\cos x - 7\sin x < 1$), so $x = \frac{3}{2}\pi + \omega$ or $\pi + \omega$.

b proceeding as in (i), $\cos(x-\alpha) = \cos\beta$, where $\tan\alpha = \frac{11}{2}$ and $R = 5\sqrt{5}$, so $\cos\beta = \dfrac{2}{\sqrt{5}}$ and so $\tan\beta = \frac{1}{2}$. Hence $\tan x = \dfrac{22 \pm 2}{4 \mp 11} = -\frac{24}{7}$ or $\frac{4}{3}$.

Notice that $\tan 2\omega = \dfrac{2\tan\omega}{1-\tan^2\omega} = -\frac{24}{7}$ so the solutions are $x = \omega$ and $x = 2\omega$, again eliminating the other two possibilities, $\omega + \pi$ and $2\omega + \pi$, by checking in the original equation.