

STEP III, 2004, Q4 MS

- 4 If circle n has centre O_n then $OO_n = \frac{r_n}{\sin \alpha}$, $OO_{n+1} = \frac{r_{n+1}}{\sin \alpha}$ and $OO_n - OO_{n+1} = r_n + r_{n+1}$.

Substituting and multiplying by $\sin \alpha$ gives $r_n - r_{n+1} = \sin \alpha (r_n + r_{n+1})$ which simplifies to the required result.

This result then implies that $r_n = \left(\frac{1 - \sin \alpha}{1 + \sin \alpha} \right)^n r_0$, so the total area is

$$S = \frac{1}{2} \pi r_0^2 + \pi \left(\left(\frac{1 - \sin \alpha}{1 + \sin \alpha} \right) r_0 \right)^2 + \pi \left(\left(\frac{1 - \sin \alpha}{1 + \sin \alpha} \right)^2 r_0 \right)^2 + \pi \left(\left(\frac{1 - \sin \alpha}{1 + \sin \alpha} \right)^3 r_0 \right)^2 + \dots$$

which is almost a geometric series with common ratio $\left(\frac{1 - \sin \alpha}{1 + \sin \alpha} \right)^2$, so

$$S = \frac{\pi r_0^2}{1 - \left(\frac{1 - \sin \alpha}{1 + \sin \alpha} \right)^2} - \frac{1}{2} \pi r_0^2 = \left(\frac{(1 + \sin \alpha)^2}{4 \sin \alpha} - \frac{1}{2} \right) \pi r_0^2 = \frac{1 + \sin^2 \alpha}{4 \sin \alpha} \pi r_0^2.$$

$$\text{Area } T \text{ of triangle } OAB = \frac{1}{2} AB \times OO_0 = \frac{r_0}{\cos \alpha} \frac{r_0}{\sin \alpha},$$

$$\text{so } \frac{S}{T} = \frac{\pi}{4} \cos \alpha (1 + \sin^2 \alpha) = \frac{\pi}{4} \cos \alpha (2 - \cos^2 \alpha).$$

By differentiation, the maximum $\frac{S}{T}$ occurs where $2 - 3 \cos^2 \alpha = 0$ (not $\sin \alpha = 0$) and equals

$$\frac{\pi}{4} \sqrt{\frac{2}{3}} \left(2 - \frac{2}{3} \right) = \frac{\pi}{3} \sqrt{\frac{2}{3}} > \sqrt{\frac{2}{3}} = \sqrt{\frac{16}{24}} > \sqrt{\frac{16}{25}} = \frac{4}{5}.$$



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