STEP III, 2004, Q4 MS

4 If circle n has centre O_n then $OO_n = \dfrac{r_n}{\sin\alpha}$, $OO_{n+1} = \dfrac{r_{n+1}}{\sin\alpha}$ and $OO_n - OO_{n+1} = r_n + r_{n+1}$.

Substituting and multiplying by $\sin\alpha$ gives $r_n - r_{n+1} = \sin\alpha(r_n + r_{n+1})$ which simplifies to the required result.

This result then implies that $r_n = \left(\dfrac{1-\sin\alpha}{1+\sin\alpha}\right)^n r_0$, so the total area is

$$S = \frac{1}{2}\pi r_0^2 + \pi\left(\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right) r_0\right)^2 + \pi\left(\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^2 r_0\right)^2 + \pi\left(\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^3 r_0\right)^2 + \dots$$

which is almost a geometric series with common ratio $\left(\dfrac{1-\sin\alpha}{1+\sin\alpha}\right)^2$, so

$$S = \frac{\pi r_0^2}{1-\left(\dfrac{1-\sin\alpha}{1+\sin\alpha}\right)^2} - \frac{1}{2}\pi r_0^2 = \left(\frac{(1+\sin\alpha)^2}{4\sin\alpha} - \frac{1}{2}\right)\pi r_0^2 = \frac{1+\sin^2\alpha}{4\sin\alpha}\pi r_0^2.$$

Area T of triangle $OAB = \frac{1}{2}AB \times OO_0 = \dfrac{r_0}{\cos\alpha}\dfrac{r_0}{\sin\alpha}$,

so $\dfrac{S}{T} = \dfrac{\pi}{4}\cos\alpha\,(1+\sin^2\alpha) = \dfrac{\pi}{4}\cos\alpha\,(2-\cos^2\alpha)$.

By differentiation, the maximum $\dfrac{S}{T}$ occurs where $2 - 3\cos^2\alpha = 0$ (not $\sin\alpha = 0$) and equals

$$\frac{\pi}{4}\sqrt{\frac{2}{3}}\left(2-\frac{2}{3}\right) = \frac{\pi}{3}\sqrt{\frac{2}{3}} > \sqrt{\frac{2}{3}} = \sqrt{\frac{16}{24}} > \sqrt{\frac{16}{25}} = \frac{4}{5}.$$