STEP III, 2004, Q4

4 The triangle OAB is isosceles, with $OA = OB$ and angle $AOB = 2\alpha$ where $0 < \alpha < \frac{\pi}{2}$. The semi-circle C_0 has its centre at the midpoint of the base AB of the triangle, and the sides OA and OB of the triangle are both tangent to the semi-circle. $C_1, C_2, C_3, \ldots$ are circles such that C_n is tangent to C_{n-1} and to sides OA and OB of the triangle.

Let r_n be the radius of C_n. Show that

$$\frac{r_{n+1}}{r_n} = \frac{1 - \sin\alpha}{1 + \sin\alpha}.$$

Let S be the total area of the semi-circle C_0 and the circles C_1, C_2, C_3, ... Show that

$$S = \frac{1 + \sin^2\alpha}{4\sin\alpha}\pi r_0^2.$$

Show that there are values of α for which S is more than four fifths of the area of triangle OAB.