STEP III, 2004, Q3 MS

3 The sketch should show a curve with increasing gradient: because the gradient is increasing, the curve lies below the chord joining $(a, \mathrm{f}(a))$ and $(b, \mathrm{f}(b))$ and above the tangent to the curve at $\left(\frac{a+b}{2}, \mathrm{f}\left(\frac{a+b}{2}\right)\right)$. The illustration is clearer if $\mathrm{f}(x) > 0$ for $a \leqslant x \leqslant b$: then the area of the trapezium cut off by the chord and the lines $x = a$, $x = b$ and $y = 0$, which is $(b-a)\frac{\mathrm{f}(a)+\mathrm{f}(b)}{2}$, is larger than the area represented by the integral and the area of the trapezium cut off by the tangent and the lines $x = a$, $x = b$ and $y = 0$, which is $(b-a)\mathrm{f}\left(\frac{a+b}{2}\right)$, is smaller than the area represented by the integral.

Choose $\mathrm{f}(x) = \frac{1}{x^2}$, checking that this has $\mathrm{f}''(x) > 0$, $a = n-1$ and $b = n$ to get the quoted result.

Take the sum from $n = 2$ to ∞ of each term in the inequality: the left hand sum is directly as quoted; in the middle sum, you need to notice that it telescopes, so that all the terms except the first cancel in pairs; in the right hand sum, each reciprocal square occurs twice, cancelling the factor of $\frac{1}{2}$, except the first.

For the next part, observe that $\frac{1}{(n+1)^2} < \frac{1}{n^2}$, so $\frac{1}{2}\left(\frac{1}{n^2} + \frac{1}{(n+1)^2}\right) < \frac{1}{2}\left(\frac{1}{n^2} + \frac{1}{n^2}\right) = \frac{1}{n^2}$.

Finally, combine the two previous results to get

$$2\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right) < 1 < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots,$$

so that if $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$, then $2\left(S - 1 - \frac{1}{2^2}\right) < 1 < S - \frac{1}{2}$; rearranging these inequalities gives the required bounds on S.