STEP III, 2004, Q3

3 Given that $\mathrm{f}''(x) > 0$ when $a \leqslant x \leqslant b$, explain with the aid of a sketch why

$$(b-a)\,\mathrm{f}\left(\frac{a+b}{2}\right) < \int_a^b \mathrm{f}(x)\,\mathrm{d}x < (b-a)\,\frac{\mathrm{f}(a)+\mathrm{f}(b)}{2}\,.$$

By choosing suitable a, b and $\mathrm{f}(x)$, show that

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{n^2} + \frac{1}{(n-1)^2}\right),$$

where n is an integer greater than 1.

Deduce that

$$4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right).$$

Show that

$$\frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots\right) < \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

and hence show that

$$\frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}\,.$$