STEP III, 2004, Q2 MS

2 (i) Inspection of the denominator shows that the vertical asymptotes are at $x = 0$, $x = 4$, and the third term in $f(x)$ tends to zero as $|x| \to \infty$, so the oblique asymptote is just $y = x - 4$.

The oblique asymptote meets the curve when $\dfrac{16(2x+1)^2}{x^2(x-4)} = 0$ or $(2x+1)^2 = 0$, hence there is a double root at $x = -\frac{1}{2}$ and hence the asymptote touches rather than crosses the curve at $(-\frac{1}{2}, -\frac{9}{2})$, so is a tangent there.

(ii) $f(x) = 0$ when $x^2(x-4)^2 - 16(2x+1)^2 = 0$.

The left hand side of this equation is a difference of two squares, so factorises to give $(x(x-4) - 4(2x+1))(x(x-4) + 4(2x+1)) = 0$; that is, $(x^2 - 12x - 4)(x+2)^2 = 0$, which has a double root at $x = -2$.

(iii) On your sketch you should show:

the double root at $(-2, 0)$ — the curve has a local maximum here and touches the x-axis;

the remaining roots (solutions of $x^2 - 12x - 4 = 0$) at $x = 6 \pm 2\sqrt{10}$;

the curve approaching the oblique asymptote $y = x - 4$ from below as $x \to \infty$, approaching it from above as $x \to -\infty$ and touching it at $(-\frac{1}{2}, -\frac{9}{2})$;

$f(x) \to \infty$ as $x \to 0$ from above or below, $f(x) \to +\infty$ as $x \to 4$ from below and $f(x) \to -\infty$ as $x \to 4$ from above;

local minima at some x value with $0 < x < 4$ and with $y > 0$ and at some x value with $-2 < x < -\frac{1}{2}$ and with $-\frac{9}{2} > y > x - 4$ — note that this second minimum is not at the point of tangency with the oblique asymptote.