STEP III, 2004, Q1 MS

1 The substitution $u = \cosh x$ should suggest itself (because of the factor of $\frac{\mathrm{d}u}{\mathrm{d}x} = \sinh x$ in t numerator), and the resulting integral can be tackled by splitting the integrand into part fractions:

$$\int_0^a \frac{\sinh x}{2\cosh^2 x - 1}\,\mathrm{d}x = \int_1^{\cosh a} \frac{\mathrm{d}u}{2u^2 - 1} = \frac{1}{2}\int_1^{\cosh a} \frac{1}{\sqrt{2}u - 1} - \frac{1}{\sqrt{2}u + 1}\,\mathrm{d}u$$

$$= \frac{1}{2\sqrt{2}}\left[\ln\left(\sqrt{2}u - 1\right) - \ln\left(\sqrt{2}u + 1\right)\right]_1^{\cosh a} = \frac{1}{2\sqrt{2}}\left(\ln\left(\frac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) + \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right)$$

Similarly, substituting $u = \sinh x$, and then recognising an arctan integral:

$$\int_0^a \frac{\cosh x}{1 + 2\sinh^2 x}\,\mathrm{d}x = \int_0^{\sinh a} \frac{\mathrm{d}u}{1 + 2u^2} = \frac{1}{\sqrt{2}}\left[\arctan\left(\sqrt{2}u\right)\right]_0^{\sinh a} = \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\sinh a\right)$$

To show that

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2\sinh^2 x}\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

note that

a $\cosh^2 x = \sinh^2 x + 1$, so $2\cosh^2 x - 1 = 1 + 2\sinh^2 x$, and the integral required is t second minus the first of those calculated earlier, as $a \to \infty$.

b as $a \to \infty$, $\sinh a \to \infty$, so $\arctan\left(\sqrt{2}\sinh a\right) \to \frac{\pi}{2}$

c as $a \to \infty$, $\cosh a \to \infty$, so $\frac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1} \to 1$ and $\ln\left(\frac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) \to 0$.

Substituting $u = \mathrm{e}^x$, so that $\cosh x = \frac{1}{2}\left(u + \frac{1}{u}\right)$ and $\sinh x = \frac{1}{2}\left(u - \frac{1}{u}\right)$:

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2\sinh^2 x}\,\mathrm{d}x = \int_1^\infty \frac{\left(u + \frac{1}{u}\right) - \left(u - \frac{1}{u}\right)}{1 + \frac{1}{2}\left(u^2 - 2 + \frac{1}{u^2}\right)}\,\frac{1}{u}\,\mathrm{d}u = \int_1^\infty \frac{2}{1 + u^4}\,\mathrm{d}u$$

so $\displaystyle\int_1^\infty \frac{1}{1 + u^4}\,\mathrm{d}u = \frac{\pi}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$