

## STEP III, 2004, Q1 MS

- 1 The substitution  $u = \cosh x$  should suggest itself (because of the factor of  $\frac{du}{dx} = \sinh x$  in the numerator), and the resulting integral can be tackled by splitting the integrand into part fractions:

$$\int_0^a \frac{\sinh x}{2 \cosh^2 x - 1} dx = \int_1^{\cosh a} \frac{du}{2u^2 - 1} = \frac{1}{2} \int_1^{\cosh a} \frac{1}{\sqrt{2}u - 1} - \frac{1}{\sqrt{2}u + 1} du$$

$$= \frac{1}{2\sqrt{2}} \left[ \ln(\sqrt{2}u - 1) - \ln(\sqrt{2}u + 1) \right]_1^{\cosh a} = \frac{1}{2\sqrt{2}} \left( \ln \left( \frac{\sqrt{2} \cosh a - 1}{\sqrt{2} \cosh a + 1} \right) + \ln \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) \right)$$

Similarly, substituting  $u = \sinh x$ , and then recognising an arctan integral:

$$\int_0^a \frac{\cosh x}{1 + 2 \sinh^2 x} dx = \int_0^{\sinh a} \frac{du}{1 + 2u^2} = \frac{1}{\sqrt{2}} \left[ \arctan(\sqrt{2}u) \right]_0^{\sinh a} = \frac{1}{\sqrt{2}} \arctan(\sqrt{2} \sinh a)$$

To show that

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2 \sinh^2 x} dx = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \ln \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right).$$

note that

- a  $\cosh^2 x = \sinh^2 x + 1$ , so  $2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x$ , and the integral required is the second minus the first of those calculated earlier, as  $a \rightarrow \infty$ .
- b as  $a \rightarrow \infty$ ,  $\sinh a \rightarrow \infty$ , so  $\arctan(\sqrt{2} \sinh a) \rightarrow \frac{\pi}{2}$
- c as  $a \rightarrow \infty$ ,  $\cosh a \rightarrow \infty$ , so  $\frac{\sqrt{2} \cosh a - 1}{\sqrt{2} \cosh a + 1} \rightarrow 1$  and  $\ln \left( \frac{\sqrt{2} \cosh a - 1}{\sqrt{2} \cosh a + 1} \right) \rightarrow 0$ .

Substituting  $u = e^x$ , so that  $\cosh x = \frac{1}{2} \left( u + \frac{1}{u} \right)$  and  $\sinh x = \frac{1}{2} \left( u - \frac{1}{u} \right)$ :

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2 \sinh^2 x} dx = \int_1^\infty \frac{\left( u + \frac{1}{u} \right) - \left( u - \frac{1}{u} \right)}{1 + \frac{1}{2} \left( u^2 - 2 + \frac{1}{u^2} \right)} \frac{1}{u} du = \int_1^\infty \frac{2}{1 + u^4} du$$

$$\text{so } \int_1^\infty \frac{1}{1 + u^4} du = \frac{\pi}{4\sqrt{2}} - \frac{1}{4\sqrt{2}} \ln \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right).$$



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