STEP III, 2004, Q14 MS

14 The test is appropriate because, if H_0 were true, $\bar{x}$ would have a higher probability of being in the region stated than if H_1 were true.

Under H_0, $\bar{X}$ has a Normal distribution with mean μ and standard deviation $\dfrac{\sigma_0}{\sqrt{n}}$ so

$$\alpha = \mathrm{P}(|\bar{X} - \mu| > c) = 2\left(1 - \Phi\left(\frac{c}{\frac{\sigma_0}{\sqrt{n}}}\right)\right)$$

so $\Phi\left(\dfrac{c}{\frac{\sigma_0}{\sqrt{n}}}\right) = 1 - \dfrac{\alpha}{2}$, so $\dfrac{c}{\frac{\sigma_0}{\sqrt{n}}} = z_\alpha$ or $c = \dfrac{\sigma_0 z_\alpha}{\sqrt{n}}$.

Under H_1, $\bar{X}$ has a Normal distribution with mean μ and standard deviation $\dfrac{\sigma_1}{\sqrt{n}}$ so

$$\beta = \mathrm{P}(|\bar{X} - \mu| < c) = 1 - 2\left(1 - \Phi\left(\frac{c}{\frac{\sigma_1}{\sqrt{n}}}\right)\right) = 2\Phi\left(\frac{c}{\frac{\sigma_1}{\sqrt{n}}}\right) - 1 = 2\Phi\left(\frac{\sigma_0 z_\alpha}{\sigma_1}\right) - 1,$$

so β is independent of n.

$$\beta < 0.05 \quad \Rightarrow \quad \Phi\left(\frac{\sigma_0 z_\alpha}{\sigma_1}\right) < \frac{1 + 0.05}{2} = 0.525 \quad \Rightarrow \quad \frac{\sigma_0 z_\alpha}{\sigma_1} < 0.063.$$

$$\alpha < 0.05 \quad \Rightarrow \quad z_\alpha > 1.960.$$

For these both to hold, we must have $0.063 > \dfrac{\sigma_0 z_\alpha}{\sigma_1} > \dfrac{1.960\sigma_0}{\sigma_1}$ or $\dfrac{\sigma_1}{\sigma_0} > \dfrac{1.960}{.063} = \dfrac{280}{9} \approx 30.$