STEP III, 2004, Q13 MS

13 (i) P(a competitor drops out in round r) $= p^{r-1}(1-p)$

so P(all three drop out in round r)$=\left(p^{r-1}(1-p)\right)^3$,

so P(all three drop out in the same round) $\equiv P_3 = \sum_{r=1}^{\infty}\left(p^{r-1}(1-p)\right)^3$

This is a geometric series with common ratio p^3 and first term $(1-p)^3$

so $P_3 = \dfrac{(1-p)^3}{1-p^3}$.

(ii) The probability that a competitor survives round $r-1$ is p^{r-1}, so the probability that a competitor drops out in round $r-1$ or earlier (that is, before round r) is $1-p^{r-1}$. Therefore the probability that two competitors drop out in round r and the third earlier is $3\times\left(p^{r-1}(1-p)\right)^2\times\left(1-p^{r-1}\right)$, where the factor of three is required, because any of the three could be the one to drop out earliest.

(iii) From (ii), Pr(two drop out in same round and the third earlier) $\equiv P_2$

$= \sum_{r=2}^{\infty} 3\left(p^{r-1}(1-p)\right)^2\left(1-p^{r-1}\right) = 3(1-p)^2\sum_{r=2}^{\infty}\left(p^{2(r-1)}-p^{3(r-1)}\right)$

$=3(1-p)^2\left(\dfrac{p^2}{1-p^2}-\dfrac{p^3}{1-p^3}\right)$, summing to infinity two geometric series with first terms p^2 and p^3 and common ratios p^2 and p^3 respectively.

Pr(the grand prize is awarded) $=1-P_2-P_3$, which simplifies to $\dfrac{3p\left(1+p^2\right)}{(1+p)\left(1+p+p^2\right)}$, using the factorisation $1-p^3=(1-p)(1+p+p^2)$.