STEP III, 2004, Q12 MS

12 For C_1, we have $\mathrm{P}(0) = \dfrac{m-1}{m}$ and $\mathrm{P}(1) = \dfrac{1}{m}$, so that $\mathrm{E}[C_1] = 0 \times \dfrac{m-1}{m} + 1 \times \dfrac{1}{m} = \dfrac{1}{m}$ and

$$\mathrm{Var}[C_1] = \left(0^2 \times \frac{m-1}{m} + 1^2 \times \frac{1}{m}\right) - \left(\frac{1}{m}\right)^2 = \frac{m-1}{m^2}$$

$\mathrm{Cov}[C_1, C_2] = 1^2 \times \mathrm{P}(C_1 = C_2 = 1) - \mathrm{E}[C_1]\mathrm{E}[C_1]$ (since the other terms in the expectation of C_1C_2 are all zero). $\mathrm{P}(C_1 = C_2 = 1) = \mathrm{P}(\text{players 1 and 2 get their own shirts}) = \dfrac{1}{m}\dfrac{1}{m-1}$,

so $\mathrm{Cov}[C_1, C_2] = \dfrac{1}{m(m-1)} - \left(\dfrac{1}{m}\right)^2 = \dfrac{1}{m^2(m-1)}$

$\mathrm{E}[N] = \mathrm{E}[C_1] + \mathrm{E}[C_2] + \ldots = m \cdot \dfrac{1}{m} = 1$ and $\mathrm{Var}[N] = \mathrm{Var}[C_1] + \mathrm{Var}[C_2] + \ldots + \mathrm{Cov}[C_1, C_2] + \mathrm{Cov}[C_1, C_3] + \mathrm{Cov}[C_2, C_1] + \ldots = m \cdot \mathrm{Var}[C_1] + m(m-1) \cdot \mathrm{Cov}[C_1, C_2] = 1$.

A normal approximation with mean and standard deviation both equal to 1 is not likely to be appropriate as the approximation would give high probability to negative values of N, which are impossible. A Poisson approximation might be reasonable as mean = variance.

There are 9 arrangements where no player wears his own shirt out of 24 permutations,while the Poisson approximation to P(0), with mean 1, is e^{-1}.

The relative error is $\dfrac{\frac{9}{24} - \mathrm{e}^{-1}}{\frac{9}{24}} \approx 1 - \dfrac{800}{3 \times 272} = \dfrac{2}{102} \approx 2\%$.