STEP III, 2004, Q11 MS

11 Since z is initially, and hence always, positive, Newton's Law gives $2\ddot{x_1} = -\frac{2}{z^3}$ and $\ddot{x_2} = \frac{2}{z^3}$, so that $\ddot{z} = \ddot{x_2} - \ddot{x_1} = \frac{3}{z^3}$.

Writing this equation as $v\frac{\mathrm{d}v}{\mathrm{d}z} = \frac{3}{z^3}$ and integrating with respect to z, we have $\frac{v^2}{2} = -\frac{3}{2z^2} + c$ so $v = \pm\sqrt{2c - \frac{3}{z^2}}$ where the initial condition $v = -1$ when $z = 1$ requires the negative sign to be chosen and c=2.

Writing $v = \frac{\mathrm{d}z}{\mathrm{d}t}$ and separating the variables gives

$$\int \frac{\mathrm{d}z}{\sqrt{4 - \frac{3}{z^2}}} = -\int \mathrm{d}t \quad \text{or} \quad c - t = \int \frac{z\mathrm{d}z}{\sqrt{4z^2 - 3}} = \tfrac{1}{4}\sqrt{4z^2 - 3}$$

so that $\sqrt{4z^2 - 3} = 1 - 4t$, using the initial condition $z = 1$ at $t = 0$ to determine c.

Then $z = \sqrt{4t^2 - 2t + 1}$ as required.

Defining $w = x_2 + 2x_1$, $\ddot{w} = \ddot{x_2} + 2\ddot{x_1} = 0$ so that $\dot{w} = a$, $w = at + b$. Initially, $\dot{x_1} = 1$ and $\dot{x_2} = 0$ so $a = 2$; $x_1 = 0$ and $x_2 = 1$ so $b = 1$. This gives

$x_1 = \frac{1}{3}(w - z) = \frac{1}{3}\left(2t + 1 - \sqrt{4t^2 - 2t + 1}\right)$ and $x_2 = \frac{1}{3}(w + 2z) = \frac{1}{3}\left(2t + 1 + 2\sqrt{4t^2 - 2t + 1}\right)$.

It is worth noting, though not required by the question, that $\dot{x_1} \to \frac{1}{2}$, $\dot{x_2} \to 2$ as $t \to \infty$.