STEP III, 2004, Q10 MS

For $6a \leqslant x \leqslant 7a$, $\ddot{x} = g + 6g\dfrac{(7a-x)}{2a} - 6g\dfrac{(x-6a)}{6a} = \dfrac{4g}{a}(7a-x)$

and for $7a \leqslant x \leqslant 9a$, $\ddot{x} = g - 6g\dfrac{(x-6a)}{6a} = \dfrac{g}{a}(7a-x)$.

Notice that these both describe simple harmonic motion with $x = 7a$ as the equilibrium position so that, for $6a \leqslant x \leqslant 7a$,

$$x = 7a + A\cos\sqrt{\frac{4g}{a}}t + B\sin\sqrt{\frac{4g}{a}}t \text{ and } \dot{x} = -\sqrt{\frac{4g}{a}}A\sin\sqrt{\frac{4g}{a}}t + \sqrt{\frac{4g}{a}}B\cos\sqrt{\frac{4g}{a}}t$$

and initial conditions $x = 6a, \dot{x} = 0$ at $t = 0$ then give $A = -a, B = 0$.

Let the particle pass through $x = 7a$ at $t = t_0$; then $\sqrt{\dfrac{4g}{a}}t_0 = \dfrac{\pi}{2}$ and, at this point, $\dot{x} = \sqrt{4ga}$.

For $7a \leqslant x \leqslant 9a$, similarly $x = 7a + A\cos\sqrt{\dfrac{g}{a}}(t-t_0) + B\sin\sqrt{\dfrac{g}{a}}(t-t_0)$. The initial conditions are $x = 7a, \dot{x} = \sqrt{4ga}$ at $t = t_0$, which give $A = 0, B = 2a$.

Finally, $x = 9a$ when $\sqrt{\dfrac{g}{a}}(t-t_0) = \dfrac{\pi}{2}$; that is, when $t = \dfrac{\pi}{2}\sqrt{\dfrac{a}{4g}} + \dfrac{\pi}{2}\sqrt{\dfrac{a}{g}} = \dfrac{3\pi}{4}\sqrt{\dfrac{a}{g}}$.