STEP III, 2003 Q5

5 Find the coordinates of the turning point on the curve $y = x^2 - 2bx + c$. Sketch the curve in the case that the equation $x^2 - 2bx + c = 0$ has two distinct real roots. Use your sketch to determine necessary and sufficient conditions on b and c for the equation $x^2 - 2bx + c = 0$ to have two distinct real roots. Determine necessary and sufficient conditions on b and c for this equation to have two distinct positive roots.

Find the coordinates of the turning points on the curve $y = x^3 - 3b^2x + c$ (with $b > 0$) and hence determine necessary and sufficient conditions on b and c for the equation $x^3 - 3b^2x + c = 0$ to have three distinct real roots. Determine necessary and sufficient conditions on a, b and c for the equation $(x - a)^3 - 3b^2(x - a) + c = 0$ to have three distinct positive roots.

Show that the equation $2x^3 - 9x^2 + 7x - 1 = 0$ has three distinct positive roots.