STEP III, 2002 Q12

12 In a game, a player tosses a biased coin repeatedly until two successive tails occur, when the game terminates. For each head which occurs the player wins £1. If E is the expected number of tosses of the coin in the course of a game, and p is the probability of a head, explain why

$$E = p\,(1 + E) + (1 - p)\,p\,(2 + E) + 2\,(1 - p)^2\ ,$$

and hence determine E in terms of p. Find also, in terms of p, the expected winnings in the course of a game.

A second game is played, with the same rules, except that the player continues to toss the coin until r successive tails occur. Show that the expected number of tosses in the course of a game is given by the expression $\dfrac{1 - q^r}{pq^r}$, where $q = 1 - p$.