STEP III, 2000 Q8

8 The sequence a_n is defined by $a_0 = 1$, $a_1 = 1$, and

$$a_n = \frac{1 + a_{n-1}^2}{a_{n-2}} \qquad (n \geqslant 2).$$

Prove by induction that

$$a_n = 3a_{n-1} - a_{n-2} \qquad (n \geqslant 2).$$

Hence show that

$$a_n = \frac{\alpha^{2n-1} + \alpha^{-(2n-1)}}{\sqrt{5}} \qquad (n \geqslant 1),$$

where $\alpha = \dfrac{1 + \sqrt{5}}{2}$.