STEP III, 2000 Q7

7 Given that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{r!} + \cdots ,$$

use the binomial theorem to show that

$$\left(1 + \frac{1}{n}\right)^n < e$$

for any positive integer n.

The product $P(n)$ is defined, for any positive integer n, by

$$P(n) = \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \ldots \cdot \frac{2^n + 1}{2^n}.$$

Use the arithmetic-geometric mean inequality,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geqslant (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n}} ,$$

to show that $P(n) < e$ for all n .

Explain briefly why $P(n)$ tends to a limit as $n \to \infty$. Show that this limit, L, satisfies $2 < L \leqslant e$.