# STEP III, 2000 Q5

**5** Given two non-zero vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ we define $\Delta(\mathbf{a}, \mathbf{b})$ by $\Delta(\mathbf{a}, \mathbf{b}) = a_1 b_2 - a_2 b_1$.

Let $A$, $B$ and $C$ be points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, no two of which are parallel. Let $P$, $Q$ and $R$ be points with position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, respectively, none of which are parallel.

(i) Show that there exists a $2 \times 2$ matrix $\mathbf{M}$ such that $P$ and $Q$ are the images of $A$ and $B$ under the transformation represented by $\mathbf{M}$.

(ii) Show that $\Delta(\mathbf{a}, \mathbf{b})\,\mathbf{c} + \boldsymbol{\Delta}(\mathbf{c}, \mathbf{a})\,\mathbf{b} + \boldsymbol{\Delta}(\mathbf{b}, \mathbf{c})\,\mathbf{a} = \mathbf{0}$.

Hence, or otherwise, prove that a necessary and sufficient condition for the points $P$, $Q$, and $R$ to be the images of points $A$, $B$ and $C$ under the transformation represented by some $2 \times 2$ matrix $\mathbf{M}$ is that

$$\Delta(\mathbf{a}, \mathbf{b}) : \Delta(\mathbf{b}, \mathbf{c}) : \Delta(\mathbf{c}, \mathbf{a}) = \Delta(\mathbf{p}, \mathbf{q}) : \Delta(\mathbf{q}, \mathbf{r}) : \Delta(\mathbf{r}, \mathbf{p})\,.$$