STEP III, 2000 Q3

3 Given that $\alpha = \mathrm{e}^{i\frac{\pi}{3}}$, prove that $1 + \alpha^2 = \alpha$.

A triangle in the Argand plane has vertices A, B, and C represented by the complex numbers p, $q\alpha^2$ and $-r\alpha$ respectively, where p, q and r are positive real numbers. Sketch the triangle ABC.

Three equilateral triangles ABL, BCM and CAN (each lettered clockwise) are erected on sides AB, BC and CA respectively. Show that the complex number representing N is $(1 - \alpha)p - \alpha^2 r$ and find similar expressions for the complex numbers representing L and M.

Show that lines LC, MA and NB all meet at the origin, and that these three line segments have the common length $p + q + r$.