STEP III, 2000 Q14

14 The random variable X takes only the values x_1 and x_2 (where $x_1 \neq x_2$), and the random variable Y takes only the values y_1 and y_2 (where $y_1 \neq y_2$). Their joint distribution is given by

$$\mathrm{P}(X = x_1, Y = y_1) = a\ ; \quad \mathrm{P}(X = x_1, Y = y_2) = q - a\ ; \quad \mathrm{P}(X = x_2, Y = y_1) = p - a\ .$$

Show that if $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$ then

$$(a - pq)(x_1 - x_2)(y_1 - y_2) = 0.$$

Hence show that two random variables each taking only two distinct values are independent if $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$.

Give a joint distribution for two random variables A and B, each taking the three values -1, 0 and 1 with probability $\frac{1}{3}$, which have $\mathrm{E}(AB) = \mathrm{E}(A)\mathrm{E}(B)$, but which are not independent.