STEP III, 2000 Q12

12 In a lottery, any one of N numbers, where N is large, is chosen at random and independently for each player by machine. Each week there are $2N$ players and one winning number is drawn. Write down an exact expression for the probability that there are three or fewer winners in a week, given that you hold a winning ticket that week. Using the fact that

$$\left(1-\frac{a}{n}\right)^n \approx \mathrm{e}^{-a}$$

for n much larger than a, or otherwise, show that this probability is approximately $\frac{2}{3}$.

Discuss briefly whether this probability would increase or decrease if the numbers were chosen by the players.

Show that the expected number of winners in a week, given that you hold a winning ticket that week, is $3 - N^{-1}$.