STEP III, 1999, Q8

8 The function $y(x)$ is defined for $x \geqslant 0$ and satisfies the conditions

$$y = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \quad \text{at } x = 0.$$

When x is in the range $2(n-1)\pi < x < 2n\pi$, where n is a positive integer, $y(t)$ satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + n^2y = 0.$$

Both y and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ are continuous at $x = 2n\pi$ for $n = 0,\ 1,\ 2,\ \ldots$.

(i) Find $y(x)$ for $0 \leqslant x \leqslant 2\pi$.

(ii) Show that $y(x) = \frac{1}{2}\sin 2x$ for $2\pi \leqslant x \leqslant 4\pi$, and find $y(x)$ for all $x \geqslant 0$.

(iii) Show that

$$\int_0^\infty y^2 \mathrm{d}x = \pi \sum_{n=1}^{\infty} \frac{1}{n^2}\,.$$