STEP III, 1998, Q8

8 (i) Show that the line $\mathbf{r} = \mathbf{b} + \lambda\mathbf{m}$, where $\mathbf{m}$ is a unit vector, intersects the sphere $\mathbf{r}.\mathbf{r} = a^2$ at two points if

$$a^2 > \mathbf{b}.\mathbf{b} - (\mathbf{b}.\mathbf{m})^2 .$$

Write down the corresponding condition for there to be precisely one point of intersection. If this point has position vector $\mathbf{p}$, show that $\mathbf{m}.\mathbf{p} = 0$.

(ii) Now consider a second sphere of radius a and a plane perpendicular to a unit vector $\mathbf{n}$. The centre of the sphere has position vector $\mathbf{d}$ and the minimum distance from the origin to the plane is l. What is the condition for the plane to be tangential to this second sphere?

(iii) Show that the first and second spheres intersect at right angles (*i.e.* the two radii to each point of intersection are perpendicular) if

$$\mathbf{d}.\mathbf{d} = 2a^2 .$$