STEP III, 1998, Q5

5 The exponential of a square matrix $\mathbf{A}$ is defined to be

$$\exp(\mathbf{A}) = \sum_{r=0}^{\infty} \frac{1}{r!}\mathbf{A}^r \,,$$

where $\mathbf{A}^0 = \mathbf{I}$ and $\mathbf{I}$ is the identity matrix.

Let

$$\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Show that $\mathbf{M}^2 = -\mathbf{I}$ and hence express $\exp(\theta\mathbf{M})$ as a single 2×2 matrix, where θ is a real number. Explain the geometrical significance of $\exp(\theta\mathbf{M})$.

Let

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

Express similarly $\exp(s\mathbf{N})$, where s is a real number, and explain the geometrical significance of $\exp(s\mathbf{N})$.

For which values of θ does

$$\exp(s\mathbf{N})\ \exp(\theta\mathbf{M})\ = \exp(\theta\mathbf{M})\ \exp(s\mathbf{N})$$

for all s? Interpret this fact geometrically.