STEP III, 1998, Q11

11 Consider a simple pendulum of length l and angular displacement θ, which is **not** assumed to be small. Show that

$$\frac{1}{2}l\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = g(\cos\theta - \cos\gamma)\,,$$

where γ is the maximum value of θ. Show also that the period P is given by

$$P = 2\sqrt{\frac{l}{g}}\int_0^{\gamma}\left(\sin^2(\gamma/2) - \sin^2(\theta/2)\right)^{-\frac{1}{2}}\,\mathrm{d}\theta\,.$$

By using the substitution $\sin(\theta/2) = \sin(\gamma/2)\sin\phi$, and then finding an approximate expression for the integrand using the binomial expansion, show that for small values of γ the period is approximately

$$2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\gamma^2}{16}\right)\,.$$