STEP II, 2024, Q8 MS

Question	Answer	Mark
8 i	$\left(\sqrt{x_n}-\sqrt{y_n}\right)^2 = 2a(x_n, y_n) - 2g(x_n, y_n)$ $= 2(x_{n+1} - y_{n+1})$	M1
	So $x_{n+1} - y_{n+1} \geq 0$ for $n \geq 0$	A1
	$y_0 < x_0$ is given Suppose that $y_k < x_k$: $\left(\sqrt{x_k}-\sqrt{y_k}\right)^2 > 0$ and so $x_{k+1} - y_{k+1} > 0$	
	Hence, by induction, $y_n < x_n$ for $n \geq 0$	A1
		[3]
	$y_{n+1} = \sqrt{x_n}\sqrt{y_n} > \sqrt{y_n}\sqrt{y_n} = y_n$	B1
	$x_{n+1} = \frac{1}{2}(x_n + y_n) < \frac{1}{2}(x_n + x_n) = x_n$	B1
	$y_n < x_n < x_0$ for $n \geq 0$, so the sequence is bounded above.	B1
	As shown above the sequence is increasing, so the result given at the start of the question applies. There is a value M such that $y_n \to M$ as $n \to \infty$	B1
		[4]
	$x_{n+1} - y_{n+1} = \frac{1}{2}\left(\sqrt{x_n}-\sqrt{y_n}\right)^2$ $< \frac{1}{2}\left(\sqrt{x_n}-\sqrt{y_n}\right)\left(\sqrt{x_n}+\sqrt{y_n}\right)$	M1
	$= \frac{1}{2}(x_n - y_n)$	
	$x_{n+1} - y_{n+1} = \frac{1}{2}\left(\sqrt{x_n}-\sqrt{y_n}\right)^2 > 0$, since $x_n \neq y_n$ for any value of n. Therefore $0 < x_{n+1} - y_{n+1} < \frac{1}{2}(x_n - y_n)$	A1
	Hence $x_n - y_n \to 0$ as $n \to \infty$	E1
	So $x_n \to M$ as $n \to \infty$	E1
		[4]

Question	Answer	Mark
8 ii	$\frac{dt}{dx}=\frac{1}{2}\left(1+\frac{pq}{x^2}\right)$	M1
	Limits: As $x\to 0$, $t\to -\infty$ As $x\to\infty$, $t\to\infty$	E1
	$\frac{1}{4}(p+q)^2+\frac{1}{4}\left(x-\frac{pq}{x}\right)^2=\frac{1}{4x^2}(x^4+(p^2+q^2)x^2+p^2q^2)$	M1
	$=\frac{1}{4x^2}(x^2+p^2)(x^2+q^2)$	
	$pq+\frac{1}{4}\left(x-\frac{pq}{x}\right)^2=\frac{1}{4x^2}(x^4+2pqx^2+p^2q^2)$ $=\frac{1}{4x^2}(x^2+pq)^2$	A1
	So the integral becomes: $2\int_0^\infty \frac{1}{\sqrt{(x^2+p^2)(x^2+q^2)}}dx=2I(p,q)$	A1
	Since the original integrand was an even function it is also equal to $2I\big(a(p,q),g(p,q)\big)$	E1
		[6]
	$I(x_0,y_0)=I(x_1,y_1)=\cdots=\int_0^\infty \frac{1}{x^2+M^2}dx$	M1
	$=\left[\frac{1}{M}\arctan\left(\frac{x}{M}\right)\right]_0^\infty$	A1
	$=\frac{\pi}{2M}$	A1
		[3]