STEP II, 2024, Q8

8 In this question, the following theorem may be used without proof.

Let u_1, u_2, ... be a sequence of real numbers. If the sequence is

- *bounded above, so $u_n \leqslant b$ for all n, where b is some fixed number*
- *and increasing, so $u_n \leqslant u_{n+1}$ for all n*

then there is a number $L \leqslant b$ such that $u_n \to L$ as $n \to \infty$.

For positive real numbers x and y, define $\mathrm{a}(x, y) = \frac{1}{2}(x + y)$ and $\mathrm{g}(x, y) = \sqrt{xy}$.
Let x_0 and y_0 be two positive real numbers with $y_0 < x_0$ and define, for $n \geqslant 0$

$$x_{n+1} = \mathrm{a}(x_n, y_n)\,,$$
$$y_{n+1} = \mathrm{g}(x_n, y_n)\,.$$

(i) By considering $\left(\sqrt{x_n} - \sqrt{y_n}\right)^2$, show that $y_{n+1} < x_{n+1}$, for $n \geqslant 0$. Show further that, for $n \geqslant 0$

- $x_{n+1} < x_n$
- $y_n < y_{n+1}$.

Deduce that there is a value M such that $y_n \to M$ as $n \to \infty$.

Show that $0 < x_{n+1} - y_{n+1} < \frac{1}{2}(x_n - y_n)$ and hence that $x_n - y_n \to 0$ as $n \to \infty$.

Explain why x_n also tends to M as $n \to \infty$.

(ii) Let

$$\mathrm{I}(p, q) = \int_0^\infty \frac{1}{\sqrt{(p^2 + x^2)(q^2 + x^2)}}\, \mathrm{d}x,$$

where p and q are positive real numbers with $q < p$.

Show, using the substitution $t = \frac{1}{2}\left(x - \frac{pq}{x}\right)$ in the integral

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\left(\frac{1}{4}(p + q)^2 + t^2\right)(pq + t^2)}}\, \mathrm{d}t,$$

that

$$\mathrm{I}(p, q) = \mathrm{I}\big(\mathrm{a}(p, q), \mathrm{g}(p, q)\big).$$

Hence evaluate $\mathrm{I}(x_0, y_0)$ in terms of M.