

STEP II, 2024, Q8

8 In this question, the following theorem may be used without proof.

Let u_1, u_2, \dots be a sequence of real numbers. If the sequence is

- bounded above, so $u_n \leq b$ for all n , where b is some fixed number
- and increasing, so $u_n \leq u_{n+1}$ for all n

then there is a number $L \leq b$ such that $u_n \rightarrow L$ as $n \rightarrow \infty$.

For positive real numbers x and y , define $a(x, y) = \frac{1}{2}(x + y)$ and $g(x, y) = \sqrt{xy}$.

Let x_0 and y_0 be two positive real numbers with $y_0 < x_0$ and define, for $n \geq 0$

$$\begin{aligned}x_{n+1} &= a(x_n, y_n), \\y_{n+1} &= g(x_n, y_n).\end{aligned}$$

(i) By considering $(\sqrt{x_n} - \sqrt{y_n})^2$, show that $y_{n+1} < x_{n+1}$, for $n \geq 0$. Show further that, for $n \geq 0$

- $x_{n+1} < x_n$
- $y_n < y_{n+1}$.

Deduce that there is a value M such that $y_n \rightarrow M$ as $n \rightarrow \infty$.

Show that $0 < x_{n+1} - y_{n+1} < \frac{1}{2}(x_n - y_n)$ and hence that $x_n - y_n \rightarrow 0$ as $n \rightarrow \infty$.

Explain why x_n also tends to M as $n \rightarrow \infty$.

(ii) Let

$$I(p, q) = \int_0^{\infty} \frac{1}{\sqrt{(p^2 + x^2)(q^2 + x^2)}} dx,$$

where p and q are positive real numbers with $q < p$.

Show, using the substitution $t = \frac{1}{2}\left(x - \frac{pq}{x}\right)$ in the integral

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\left(\frac{1}{4}(p+q)^2 + t^2\right)(pq + t^2)}} dt,$$

that

$$I(p, q) = I(a(p, q), g(p, q)).$$

Hence evaluate $I(x_0, y_0)$ in terms of M .



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