STEP II, 2024, Q7 MS

Question	Answer	Mark
7 i	Circles with centres at (1,0) and (-1,0),	B1
	both with radius 1	B1
		[2]
ii a	$y = k$ meets the curve when $(x^2+k^2-2x)(x^2+k^2+2x) = \frac{1}{16}$	
	$(x^2+k^2-2x)(x^2+k^2+2x) = (x^2+k^2)^2 - 4x^2$	M1
	So $(x^2+k^2)^2 - 4x^2 - \frac{1}{16} = 0$ $x^4 + 2x^2(k^2-2) + k^4 - \frac{1}{16} = 0$	A1
	$(x^2+k^2-2)^2 + 4k^2 - \frac{65}{16} = 0$ $x^2 = 2 - k^2 \pm \sqrt{\frac{65}{16} - 4k^2}$	M1 A1
	If $k^2 > \frac{65}{64}$ there will be no roots If $k^2 = \frac{65}{64}$ there will be two roots	B1
	If $k^2 < \frac{65}{64}$, the smaller of the two values of x^2 is $2 - k^2 - \sqrt{\frac{65}{16} - 4k^2}$	M1
	$2 - k^2 - \sqrt{\frac{65}{16} - 4k^2} = 0$ when $(2-k^2)^2 = \frac{65}{16} - 4k^2$	
	$k^4 = \frac{1}{16}$ So there will be three roots if $k^2 = \frac{1}{4}$	A1
	There will be two roots if $0 \le k^2 < \frac{1}{4}$ There will be four roots if $\frac{1}{4} < k^2 < \frac{65}{64}$	A1
		[8]
ii b	Greatest possible y-coordinate is when $k^2 = \frac{65}{64}$	M1
	So $x^2 = 2 - k^2 = \frac{63}{64} < 1$ So these points are closer to the y-axis than those on C_1	A1
		[2]
ii c	If both expressions are negative, then the distance from (x, y) to the points $(1,0)$ and $(-1,0)$ would both be less than 1.	E1
	But the shortest distance between $(1,0)$ and $(-1,0)$ is 2, so this is not possible. Therefore, it is not possible for both expressions to be negative.	E1
	Since the product of the two expression is positive and they are not both negative, they must both be positive. Therefore, the distance between the point (x, y) and each of the points $(1,0)$ and $(-1,0)$ must be greater than 1, so the curve C_2 lies entirely outside the circle C_1	E1
		[3]

Question	Answer	Mark
7 ii d	Continuous curve outside the two circles of C_1	G1
	Symmetrical under reflection in x and y axes.	G1
	Intersections with x-axis at $x = \pm\frac{1}{2}\sqrt{8+\sqrt{65}}$	G1
	Intersections with y-axis at $y = \pm\frac{1}{2}$	G1
	Maxima and minima at $\left(\pm\frac{1}{8}\sqrt{63}, \pm\frac{1}{8}\sqrt{65}\right)$	G1
		[5]