STEP II, 2024, Q6

6 *In this question, you need not consider issues of convergence.*

(i) The sequence T_n, for $n = 0, 1, 2, \ldots$, is defined by $T_0 = 1$ and, for $n \geqslant 1$, by

$$T_n = \frac{2n-1}{2n} T_{n-1}.$$

Prove by induction that

$$T_n = \frac{1}{2^{2n}} \binom{2n}{n},$$

for $n = 0, 1, 2, \ldots$.

[Note that $\binom{0}{0} = 1$.]

(ii) Show that in the binomial series for $(1-x)^{-\frac{1}{2}}$,

$$(1-x)^{-\frac{1}{2}} = \sum_{r=0}^{\infty} a_r x^r,$$

successive coefficients are related by

$$a_r = \frac{2r-1}{2r} a_{r-1}$$

for $r = 1, 2, \ldots$.

Hence prove that $a_r = T_r$ for all $r = 0, 1, 2, \ldots$.

(iii) Let b_r be the coefficient of x^r in the binomial series for $(1-x)^{-\frac{3}{2}}$, so that

$$(1-x)^{-\frac{3}{2}} = \sum_{r=0}^{\infty} b_r x^r.$$

By considering $\dfrac{b_r}{a_r}$, find an expression involving a binomial coefficient for b_r, for $r = 0, 1, 2, \ldots$.

(iv) By considering the product of the binomial series for $(1-x)^{-\frac{1}{2}}$ and $(1-x)^{-1}$, prove that

$$\frac{(2n+1)}{2^{2n}} \binom{2n}{n} = \sum_{r=0}^{n} \frac{1}{2^{2r}} \binom{2r}{r},$$

for $n = 1, 2, \ldots$.