STEP II, 2024, Q4 MS

Question	Answer	Mark
4 i a	$\boldsymbol{b}$ is a linear combination of $\boldsymbol{x}$ and $\boldsymbol{y}$, so it must lie in the plane OXY	B1
	$\boldsymbol{b}\cdot\boldsymbol{x} = (\lvert\boldsymbol{x}\rvert\boldsymbol{y} + \lvert\boldsymbol{y}\rvert\boldsymbol{x})\cdot\boldsymbol{x} = \lvert\boldsymbol{x}\rvert\boldsymbol{y}\cdot\boldsymbol{x} + \lvert\boldsymbol{y}\rvert\lvert\boldsymbol{x}\rvert^2$	M1
	If θ is the angle between $\boldsymbol{x}$ and $\boldsymbol{b}$, then $\cos\theta = \dfrac{\boldsymbol{b}\cdot\boldsymbol{x}}{\lvert\boldsymbol{b}\rvert\lvert\boldsymbol{x}\rvert} = \dfrac{\boldsymbol{x}\cdot\boldsymbol{y} + \lvert\boldsymbol{x}\rvert\lvert\boldsymbol{y}\rvert}{\lvert\boldsymbol{b}\rvert}$	M1
	Similarly, $\dfrac{\boldsymbol{b}\cdot\boldsymbol{y}}{\lvert\boldsymbol{b}\rvert\lvert\boldsymbol{y}\rvert} = \dfrac{\boldsymbol{x}\cdot\boldsymbol{y} + \lvert\boldsymbol{x}\rvert\lvert\boldsymbol{y}\rvert}{\lvert\boldsymbol{b}\rvert}$ so the angle between $\boldsymbol{b}$ and $\boldsymbol{y}$ is also θ.	A1
	Since $\boldsymbol{x}\cdot\boldsymbol{y} + \lvert\boldsymbol{x}\rvert\lvert\boldsymbol{y}\rvert > 0$, $\cos\theta > 0$ and so the angle is less than $90°$	E1
	A sketch to indicate why any other bisecting vector is a positive multiple of this.	E1
		[6]
i b	The vector $\overrightarrow{XB} = \lambda\boldsymbol{b} - \boldsymbol{x}$ must be parallel to the vector $\overrightarrow{XY} = \boldsymbol{y} - \boldsymbol{x}$.	
	For some μ: $\lambda\boldsymbol{b} - \boldsymbol{x} = \mu(\boldsymbol{y} - \boldsymbol{x})$	M1
	$\lambda(\lvert\boldsymbol{x}\rvert\boldsymbol{y} + \lvert\boldsymbol{y}\rvert\boldsymbol{x}) - \boldsymbol{x} = \mu(\boldsymbol{y} - \boldsymbol{x})$ $(\lambda\lvert\boldsymbol{y}\rvert + \mu - 1)\boldsymbol{x} = (\mu - \lambda\lvert\boldsymbol{x}\rvert)\boldsymbol{y}$	A1
	Since $\boldsymbol{x}$ and $\boldsymbol{y}$ are not parallel: $\lambda\lvert\boldsymbol{y}\rvert + \mu - 1 = 0$ $\mu - \lambda\lvert\boldsymbol{x}\rvert = 0$	E1
	$\lambda = \dfrac{1}{\lvert\boldsymbol{x}\rvert + \lvert\boldsymbol{y}\rvert}$	M1
	$\mu = \dfrac{\lvert\boldsymbol{x}\rvert}{\lvert\boldsymbol{x}\rvert + \lvert\boldsymbol{y}\rvert}$	M1
	So B divides XY in the ratio $\lvert\boldsymbol{x}\rvert : \lvert\boldsymbol{y}\rvert$	A1
		[6]
i c	If OB is perpendicular to XY: $\boldsymbol{b}\cdot(\boldsymbol{y} - \boldsymbol{x}) = 0$ $\lvert\boldsymbol{x}\rvert\lvert\boldsymbol{y}\rvert^2 + \lvert\boldsymbol{y}\rvert\boldsymbol{x}\cdot\boldsymbol{y} - \lvert\boldsymbol{x}\rvert\boldsymbol{y}\cdot\boldsymbol{x} - \lvert\boldsymbol{y}\rvert\lvert\boldsymbol{x}\rvert^2 = 0$ $(\lvert\boldsymbol{y}\rvert - \lvert\boldsymbol{x}\rvert)(\lvert\boldsymbol{x}\rvert\lvert\boldsymbol{y}\rvert + \boldsymbol{x}\cdot\boldsymbol{y}) = 0$	M1 A1
	$\lvert\boldsymbol{x}\rvert\lvert\boldsymbol{y}\rvert + \boldsymbol{x}\cdot\boldsymbol{y} > 0$ So $\lvert\boldsymbol{x}\rvert = \lvert\boldsymbol{y}\rvert$	A1
		[3]
ii	Let $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$ be the position vectors of P, Q and R respectively.	
	The bisecting vector of POQ is $\lvert\boldsymbol{p}\rvert\boldsymbol{q} + \lvert\boldsymbol{q}\rvert\boldsymbol{p}$ The bisecting vector of QOR is $\lvert\boldsymbol{q}\rvert\boldsymbol{r} + \lvert\boldsymbol{r}\rvert\boldsymbol{q}$	
	If θ is the angle between these two vectors, then: $\cos\theta = \dfrac{(\lvert\boldsymbol{p}\rvert\boldsymbol{q} + \lvert\boldsymbol{q}\rvert\boldsymbol{p})\cdot(\lvert\boldsymbol{q}\rvert\boldsymbol{r} + \lvert\boldsymbol{r}\rvert\boldsymbol{q})}{\lvert\lvert\boldsymbol{p}\rvert\boldsymbol{q} + \lvert\boldsymbol{q}\rvert\boldsymbol{p}\rvert\,\lvert\lvert\boldsymbol{q}\rvert\boldsymbol{r} + \lvert\boldsymbol{r}\rvert\boldsymbol{q}\rvert}$	M1
	$= \dfrac{\lvert\boldsymbol{p}\rvert\lvert\boldsymbol{q}\rvert\boldsymbol{q}\cdot\boldsymbol{r} + \lvert\boldsymbol{p}\rvert\lvert\boldsymbol{r}\rvert\lvert\boldsymbol{q}\rvert^2 + \lvert\boldsymbol{q}\rvert^2\boldsymbol{p}\cdot\boldsymbol{r} + \lvert\boldsymbol{q}\rvert\lvert\boldsymbol{r}\rvert\boldsymbol{p}\cdot\boldsymbol{q}}{\lvert\lvert\boldsymbol{p}\rvert\boldsymbol{q} + \lvert\boldsymbol{q}\rvert\boldsymbol{p}\rvert\,\lvert\lvert\boldsymbol{q}\rvert\boldsymbol{r} + \lvert\boldsymbol{r}\rvert\boldsymbol{q}\rvert}$	M1
	$\cos\theta = \dfrac{\lvert\boldsymbol{q}\rvert(\lvert\boldsymbol{p}\rvert\lvert\boldsymbol{q}\rvert\lvert\boldsymbol{r}\rvert + \lvert\boldsymbol{p}\rvert\boldsymbol{q}\cdot\boldsymbol{r} + \lvert\boldsymbol{q}\rvert\boldsymbol{p}\cdot\boldsymbol{r} + \lvert\boldsymbol{r}\rvert\boldsymbol{p}\cdot\boldsymbol{q})}{\lvert\lvert\boldsymbol{p}\rvert\boldsymbol{q} + \lvert\boldsymbol{q}\rvert\boldsymbol{p}\rvert\,\lvert\lvert\boldsymbol{q}\rvert\boldsymbol{r} + \lvert\boldsymbol{r}\rvert\boldsymbol{q}\rvert}$	A1
	$\lvert\boldsymbol{p}\rvert\lvert\boldsymbol{q}\rvert\lvert\boldsymbol{r}\rvert + \lvert\boldsymbol{p}\rvert\boldsymbol{q}\cdot\boldsymbol{r} + \lvert\boldsymbol{q}\rvert\boldsymbol{p}\cdot\boldsymbol{r} + \lvert\boldsymbol{r}\rvert\boldsymbol{p}\cdot\boldsymbol{q}$ is symmetrical in $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$.	E1
	All other factors are strictly positive, so the sign will be the same for all three angles (and so the angles are either all acute, all right angles or all obtuse).	E1
		[5]