STEP II, 2024, Q4

4 In this question, if O, C and D are non-collinear points in three dimensional space, we will call the non-zero vector $\mathbf{v}$ a *bisecting vector* for angle COD if $\mathbf{v}$ lies in the plane COD, the angle between $\mathbf{v}$ and $\overrightarrow{OC}$ is equal to the angle between $\mathbf{v}$ and $\overrightarrow{OD}$, and both angles are less than $90°$.

(i) Let O, X and Y be non-collinear points in three-dimensional space, and define $\mathbf{x} = \overrightarrow{OX}$ and $\mathbf{y} = \overrightarrow{OY}$.

Let $\mathbf{b} = |\mathbf{x}|\mathbf{y} + |\mathbf{y}|\mathbf{x}$.

(a) Show that $\mathbf{b}$ is a bisecting vector for angle XOY.

Explain, using a diagram, why any other bisecting vector for angle XOY is a positive multiple of $\mathbf{b}$.

(b) Find the value of λ such that the point B, defined by $\overrightarrow{OB} = \lambda\mathbf{b}$, lies on the line XY. Find also the ratio in which the point B divides XY.

(c) Show, in the case when OB is perpendicular to XY, that the triangle XOY is isosceles.

(ii) Let O, P, Q and R be points in three-dimensional space, no three of which are collinear. A bisecting vector is chosen for each of the angles POQ, QOR and ROP. Show that the three angles between them are either all acute, all obtuse or all right angles.