STEP II, 2024, Q2

2 *In this question, you need not consider issues of convergence.*

(i) Find the binomial series expansion of $(8+x^3)^{-1}$, valid for $|x| < 2$.

Hence show that, for each integer $m \geqslant 0$,

$$\int_0^1 \frac{x^m}{8+x^3}\,\mathrm{d}x = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+m+1}\right).$$

(ii) Show that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left(\frac{1}{3k+3}-\frac{2}{3k+2}+\frac{4}{3k+1}\right) = \int_0^1 \frac{1}{x+2}\,\mathrm{d}x\,,$$

and evaluate the integral.

(iii) Show that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left(\frac{72(2k+1)}{(3k+1)(3k+2)}\right) = \pi\sqrt{a} - \ln b\,,$$

where a and b are integers which you should determine.