# STEP II, 2024, Q11 MS

| Question | Answer | Mark |
|---|---|---|
| 11 i | $x^{\frac{1}{x}} = e^{\frac{1}{x}\ln x}$ | M1 |
| | $\frac{d}{dx}\left(x^{\frac{1}{x}}\right) = \frac{1-\ln x}{x^2}x^{\frac{1}{x}}$ | A1 |
| | $=0$ when $x=e$ | A1 |
| | $\frac{1}{x}\ln x \to 0$ as $x \to \infty$, so $x^{\frac{1}{x}} \to 1$ as $x \to \infty$ | B1 |
| | Let $x = \frac{1}{N}$, then $x^{\frac{1}{x}} = \frac{1}{N^N} \to 0$ as $N \to \infty$ / $x \to 0$ | B1 |
| | Graph showing the above and no additional turning points.<br>Both coordinates of turning point $\left(e, e^{\frac{1}{e}}\right)$ must be shown. | G1 |
| | | [6] |
| | Since the graph is decreasing for $x > e$, $3^{\frac{1}{3}} > n^{\frac{1}{n}}$ for $n > 3$ | E1 |
| | $2^{\frac{1}{2}} = 4^{\frac{1}{4}}$ and so is less than $3^{\frac{1}{3}}$, so the maximum value is $3^{\frac{1}{3}}$. | E1 |
| | | [2] |
| ii | For each group:<br>$P(\text{one test}) = (1-p)^k$ | M1 |
| | Expected number of tests is:<br>$1 \cdot (1-p)^k + (k+1)\left(1-(1-p)^k\right)$ | M1 |
| | Expected number of tests in total:<br>$r\left(k+1-k(1-p)^k\right)$<br>$= N\left(1+\frac{1}{k}-(1-p)^k\right)$ | A1 |
| | | [3] |
| iii | $N\left(1+\frac{1}{k}-(1-p)^k\right) \le N \Rightarrow \frac{1}{k} \le (1-p)^k$ | M1 |
| | $\frac{1}{1-p} \le k^{\frac{1}{k}}$ | A1 |
| | By part (i) the maximum value arises when $k=3$ | M1 |
| | and $\frac{1}{1-p} = 3^{\frac{1}{3}}$<br>$p = 1 - 3^{-\frac{1}{3}}$ | A1 |
| | $p - \frac{1}{4} = \frac{3}{4} - 3^{-\frac{1}{3}}$ | |
| | $\left(\frac{3}{4}\right)^{-3} = \frac{64}{27} < 3$, so $3^{-\frac{1}{3}} < \frac{3}{4}$ | M1 |
| | So $p - \frac{1}{4} > 0$ and $p > \frac{1}{4}$ | A1 |
| | | [6] |
| iv | The term in $p^n$ in the expansion of $(1-p)^k$ is<br>$\frac{k(k-1)\dots(k-n+1)}{n!}(-p)^n$<br>$\frac{k(k-1)\dots(k-n+1)}{n!}p^n < \frac{(kp)^n}{n!}$ | M1 |
| | If $kp$ is small:<br>$N\left(1+\frac{1}{k}-(1-p)^k\right) \approx N\left(1+\frac{1}{k}-1+kp\right)$<br>$= N\left(\frac{1}{k}+kp\right)$ | A1 |
| | If $p = 0.01$ and $k = 10$:<br>$\frac{1}{10} + 10(0.01) = \frac{1}{5}$, so the expected number of tests is approximately 20% of $N$. | E1 |
| | | [3] |


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