STEP II, 2024, Q11

11 **(i)** Sketch a graph of $y = x^{\frac{1}{x}}$ for $x > 0$, showing the location of any turning points.

Find the maximum value of $n^{\frac{1}{n}}$, where n is a positive integer.

N people are to have their blood tested for the presence or absence of an enzyme. Each person, independently of the others, has a probability p of having the enzyme present in a sample of their blood, where $0 < p < 1$. The blood test always correctly determines whether the enzyme is present or absent in a sample.

The following method is used.

- The people to be tested are split into r groups of size k, with $k > 1$ and $rk = N$.
- In every group, a sample from each person in that group is mixed into one large sample, which is then tested.
- If the enzyme is not present in the combined sample from a group, no further testing of the people in that group is needed.
- If the enzyme is present in the combined sample from a group, a second sample from each person in that group is tested separately.

(ii) Find, in terms of N, k and p, the expected number of tests.

(iii) Given that N is a multiple of 3, find the largest value of p for which it is possible to find an integer value of k such that $k > 1$ and the expected number of tests is at most N.

Show that this value of p is greater than $\frac{1}{4}$.

(iv) Show that, if pk is sufficiently small, the expected number of tests is approximately $N\left(\frac{1}{k} + pk\right)$.

In the case where $p = 0.01$, show that choosing $k = 10$ gives an expected number of tests which is only about 20% of N.