STEP II, 2023, Q8 MS

Question			Answer	Mark
8	(i)		Let vertices be numbered 1 to 4 and edges be e_{ij}, where $i < j$. Then perimeters equal is $\vert e_{12}\vert + \vert e_{23}\vert + \vert e_{13}\vert = \vert e_{12}\vert + \vert e_{24}\vert + \vert e_{14}\vert$ $= \vert e_{13}\vert + \vert e_{34}\vert + \vert e_{14}\vert$ $= \vert e_{24}\vert + \vert e_{34}\vert + \vert e_{23}\vert$	M1
			which implies $\vert e_{12}\vert + \vert e_{23}\vert + \vert e_{13}\vert + \vert e_{12}\vert + \vert e_{24}\vert + \vert e_{14}\vert$ $= \vert e_{13}\vert + \vert e_{34}\vert + \vert e_{14}\vert + \vert e_{24}\vert + \vert e_{34}\vert + \vert e_{23}\vert$ so $2\vert e_{12}\vert + (\vert e_{23}\vert + \vert e_{13}\vert + \vert e_{24}\vert + \vert e_{14}\vert)$ $= 2\vert e_{34}\vert + (\vert e_{13}\vert + \vert e_{14}\vert + \vert e_{24}\vert + \vert e_{23}\vert)$ and so $\vert e_{12}\vert = \vert e_{34}\vert$	A1
			and by permutations of this argument, all pairs of opposite sides are equal.	E1
			if $\vert e_{12}\vert = \vert e_{34}\vert$, $\vert e_{13}\vert = \vert e_{24}\vert$, $\vert e_{14}\vert = \vert e_{23}\vert$, then all perimeters are trivially equal	B1
				[4]
	(ii)		$\vert \boldsymbol{a}\vert^2 = \vert \boldsymbol{b} - \boldsymbol{c}\vert^2$	M1
			$= \vert \boldsymbol{b}\vert^2 + \vert \boldsymbol{c}\vert^2 - 2\boldsymbol{b}.\boldsymbol{c}$	A1
			From the equivalent results to **(ii)** using the other pairs of opposite sides	M1
			$\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c} = \frac{1}{2}(\vert \boldsymbol{a}\vert^2 + \vert \boldsymbol{b}\vert^2 - \vert \boldsymbol{c}\vert^2) + \frac{1}{2}(\vert \boldsymbol{a}\vert^2 + \vert \boldsymbol{c}\vert^2 - \vert \boldsymbol{b}\vert^2) = \vert \boldsymbol{a}\vert^2$	A1
				[4]
	(iii)		$16\vert \boldsymbol{a} - \boldsymbol{g}\vert^2 = \vert 3\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}\vert^2$	M1
			$= 9\vert \boldsymbol{a}\vert^2 + \vert \boldsymbol{b}\vert^2 + \vert \boldsymbol{c}\vert^2 - 6\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) + 2\boldsymbol{b}.\boldsymbol{c}$	
			$= 9\vert \boldsymbol{a}\vert^2 + \vert \boldsymbol{b}\vert^2 + \vert \boldsymbol{c}\vert^2 - 6\vert \boldsymbol{a}\vert^2 + \vert \boldsymbol{b}\vert^2 + \vert \boldsymbol{c}\vert^2 - \vert \boldsymbol{a}\vert^2$ using previous results	M1
			$= 2(\vert \boldsymbol{a}\vert^2 + \vert \boldsymbol{b}\vert^2 + \vert \boldsymbol{c}\vert^2)$	A1
			but this is symmetric in **a**, **b**, **c** so **g** equidistant from A, B and C.	A1
			$16\vert \boldsymbol{g}\vert^2 = \vert \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}\vert^2$ $= \vert \boldsymbol{a}\vert^2 + 2\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) + \vert \boldsymbol{b} + \boldsymbol{c}\vert^2$ $= \vert \boldsymbol{a}\vert^2 + 2\vert \boldsymbol{a}\vert^2 + \vert \boldsymbol{b}\vert^2 + \vert \boldsymbol{c}\vert^2 + 2\boldsymbol{b}.\boldsymbol{c}$ $= 3\vert \boldsymbol{a}\vert^2 + \vert \boldsymbol{b}\vert^2 + \vert \boldsymbol{c}\vert^2 + \vert \boldsymbol{b}\vert^2 + \vert \boldsymbol{c}\vert^2 - \vert \boldsymbol{a}\vert^2$ $= 2(\vert \boldsymbol{a}\vert^2 + \vert \boldsymbol{b}\vert^2 + \vert \boldsymbol{c}\vert^2)$ So G equidistant from O also.	B1
				[5]
	(iv)		$\vert \boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}\vert^2 = \vert \boldsymbol{a}\vert^2 + \vert \boldsymbol{b}\vert^2 + \vert \boldsymbol{c}\vert^2 - 2\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) + 2\boldsymbol{b}.\boldsymbol{c}$	M1
			$= \vert \boldsymbol{a}\vert^2 + \vert \boldsymbol{b}\vert^2 + \vert \boldsymbol{c}\vert^2 - 2\vert \boldsymbol{a}\vert^2 + \vert \boldsymbol{b}\vert^2 + \vert \boldsymbol{c}\vert^2 - \vert \boldsymbol{a}\vert^2$	M1
			$= 2(\vert \boldsymbol{b}\vert^2 + \vert \boldsymbol{c}\vert^2 - \vert \boldsymbol{a}\vert^2)$	A1
			which must be non-negative, so cos(BAC) $\geqslant 0$	M1
			and symmetry implies no angle obtuse	A1
			If e.g. BAC was a right angle, would have $\vert \boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}\vert^2 = 0$, so **a** = **b** + **c**	M1
			so O, A, B, C all in one plane, so not a tetrahedron.	A1
				[7]