STEP II, 2023, Q8

8 A tetrahedron is called isosceles if each pair of edges which do not share a vertex have equal length.

(i) Prove that a tetrahedron is isosceles if and only if all four faces have the same perimeter.

Let $OABC$ be an isosceles tetrahedron and let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

(ii) By considering the lengths of OA and BC, show that

$$2\mathbf{b}.\mathbf{c} = |\mathbf{b}|^2 + |\mathbf{c}|^2 - |\mathbf{a}|^2.$$

Show that

$$\mathbf{a}.(\mathbf{b} + \mathbf{c}) = |\mathbf{a}|^2.$$

(iii) Let G be the *centroid* of the tetrahedron, defined by $\overrightarrow{OG} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

Show that G is equidistant from all four vertices of the tetrahedron.

(iv) By considering the length of the vector $\mathbf{a} - \mathbf{b} - \mathbf{c}$, or otherwise, show that, in an isosceles tetrahedron, none of the angles between pairs of edges which share a vertex can be obtuse. Can any of them be right angles?