STEP II, 2023, Q5

5 **(i)** The sequence x_n for $n = 0, 1, 2, \ldots$ is defined by $x_0 = 1$ and by

$$x_{n+1} = \frac{x_n + 2}{x_n + 1}$$

for $n \geqslant 0$.

(a) Explain briefly why $x_n \geqslant 1$ for all n.

(b) Show that $x_{n+1}^2 - 2$ and $x_n^2 - 2$ have opposite sign, and that

$$\left|x_{n+1}^2 - 2\right| \leqslant \tfrac{1}{4}\left|x_n^2 - 2\right|.$$

(c) Show that

$$2 - 10^{-6} \leqslant x_{10}^2 \leqslant 2.$$

(ii) The sequence y_n for $n = 0, 1, 2, \ldots$ is defined by $y_0 = 1$ and by

$$y_{n+1} = \frac{y_n^2 + 2}{2y_n}$$

for $n \geqslant 0$.

(a) Show that, for $n \geqslant 0$,

$$y_{n+1} - \sqrt{2} = \frac{\left(y_n - \sqrt{2}\right)^2}{2y_n}$$

and deduce that $y_n \geqslant 1$ for $n \geqslant 0$.

(b) Show that

$$y_n - \sqrt{2} \leqslant 2\left(\frac{\sqrt{2} - 1}{2}\right)^{2^n}$$

for $n \geqslant 1$.

(c) Using the fact that

$$\sqrt{2} - 1 < \tfrac{1}{2},$$

or otherwise, show that

$$\sqrt{2} \leqslant y_{10} \leqslant \sqrt{2} + 10^{-600}.$$

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