STEP II, 2023, Q2 MS

Question			Answer	Mark
2	(i)		Let $x = \tan\alpha$. Then $y = \frac{2\tan\alpha}{1-\tan^2\alpha} = \tan 2\alpha$	
			so $z = \tan 4\alpha$, and so $\tan\alpha = \tan 8\alpha$.	E1
			giving $8\alpha = \alpha + n\pi$, or $\alpha = \frac{1}{7}n\pi$, (for $n = -3$ to 3).	M1
			Solutions are: $(0,0,0)$, $\left(\tan\left(\frac{1}{7}\alpha\right), \tan\left(\frac{2}{7}\alpha\right), \tan\left(-\frac{3}{7}\alpha\right)\right)$	B1
			and cyclic permutations of the latter	A1
			and $\left(\tan\left(-\frac{1}{7}\alpha\right), \tan\left(-\frac{2}{7}\alpha\right), \tan\left(\frac{3}{7}\alpha\right)\right)$ and its cyclic permutations	A1
				[5]
	(ii)		$\tan 3\alpha = \frac{\frac{2\tan\alpha}{1-\tan^2\alpha}+\tan\alpha}{1-\frac{2\tan^2\alpha}{1-\tan^2\alpha}}$	M1
			$= \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$	A1
			Let $x = \tan\alpha$; then $y = \tan 3\alpha$, $z = \tan 9\alpha$, so $\tan 27\alpha = \tan\alpha$	M1
			giving $26\alpha = n\pi$	A1
			which has 25 solutions with distinct values of $\tan\alpha$ because $n = 13$ does not give a possible value of $\tan\alpha$.	A1
			Checking that for each finite value of x, the denominators of y and z are defined (i.e. checking 1-3x^2 is non-zero).	E1
				[6]
	(iii)	(a)	Let $x = \cos\alpha$	M1
			the restriction on $\lvert x\rvert$ means this is a complete parametrisation of solutions	E1
			Then, using $\cos 2\alpha = 2\cos^2\alpha - 1$, $\cos 8\alpha = \cos\alpha$	M1
			so $8\alpha = \alpha + 2m\pi$, or $8\alpha = -\alpha + 2n\pi$	M1
			so $7\alpha = 2m\pi$ or $9\alpha = 2n\pi$	A1
			with 4 ($m = 0$ to 3) + 5 ($n = 0$ to 4)	M1
			– 1 (for $\alpha = 0$ twice) = 8 distinct solutions	A1
				[7]
		(b)	y quadratic, so z quartic in x, so x satisfies an octic equation	B1
			which has at most 8 roots, so there are no larger solutions.	E1
				[2]