STEP II, 2023, Q2

2 **(i)** The real numbers x, y and z satisfy the equations

$$y = \frac{2x}{1-x^2}\,,$$
$$z = \frac{2y}{1-y^2}\,,$$
$$x = \frac{2z}{1-z^2}\,.$$

Let $x = \tan\alpha$. Deduce that $y = \tan 2\alpha$ and show that $\tan\alpha = \tan 8\alpha$.

Find all solutions of the equations, giving each value of x, y and z in the form $\tan\theta$ where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(ii) Determine the number of real solutions of the simultaneous equations

$$y = \frac{3x-x^3}{1-3x^2}\,,$$
$$z = \frac{3y-y^3}{1-3y^2}\,,$$
$$x = \frac{3z-z^3}{1-3z^2}\,.$$

(iii) Consider the simultaneous equations

$$y = 2x^2 - 1\,,$$
$$z = 2y^2 - 1\,,$$
$$x = 2z^2 - 1\,.$$

(a) Determine the number of real solutions of these simultaneous equations with $|x| \leqslant 1$, $|y| \leqslant 1$, $|z| \leqslant 1$.

(b) By finding the degree of a single polynomial equation which is satisfied by x, show that all solutions of these simultaneous equations have $|x| \leqslant 1$, $|y| \leqslant 1$, $|z| \leqslant 1$.