STEP II, 2023, Q12 MS

Question			Answer	Mark
12	(i)		$P(Y \leqslant t) = P(X_i \leqslant t \text{ for all } i = 1, \ldots, n)$ $= \prod_{i=1}^{n} P(X_i \leqslant t) = [P(X_1 \leqslant t)]^n$	E1
			$= \left(\int_0^t \frac{1}{2}\sin x \, dx\right)^n = \frac{1}{2^n}(1 - \cos t)^n$	M1
			so $f_Y(t) = \frac{n \sin t}{2^n}(1 - \cos t)^{n-1}$	A1
				[3]
	(ii)		$\frac{1}{2^n}(1 - \cos m(n))^n = \frac{1}{2}$	M1
			so $m(n) = \arccos\left(1 - 2^{\frac{n-1}{n}}\right)$	A1
			which tends to π as $n \to \infty$.	A1
				[3]
	(iii)		$\mu(n) = \int_0^\pi x \frac{n}{2^n} \sin x (1 - \cos x)^{n-1} \, dx$ $= \left[x \frac{1}{2^n}(1 - \cos x)^n\right]_0^\pi - \int_0^\pi \frac{1}{2^n}(1 - \cos x)^n \, dx$	M1
			$= \left(\pi \frac{1}{2^n} 2^n - 0\right) - \int_0^\pi \frac{1}{2^n}(1 - \cos x)^n \, dx$ as required	A1
				[2]
		(a)	As $\mu(n) = \pi - \int_0^\pi \left(\frac{1 - \cos x}{2}\right)^n dx$, the integrand decreases with n throughout $(0, \pi)$	M1
			and so $\mu(n)$ increases with n	A1
				[2]
		(b)	$\mu(2) = \pi - \int_0^\pi \frac{1}{4}\left(1 - 2\cos x + \cos^2 x\right) dx$ $= \pi - \int_0^\pi \frac{1}{4}\left(1 - 2\cos x + \frac{1}{2}(1 + \cos 2x)\right) dx$	M1
			$= \pi - \left[\frac{3}{8}x - \frac{1}{2}\sin x + \frac{1}{16}\sin 2x\right]_0^\pi$	M1
			$= \frac{5}{8}\pi$	A1
			so $\cos^2(\mu(2)) = \frac{1}{2}\left(1 + \cos\frac{5}{4}\pi\right) = \frac{1}{4}(2 - \sqrt{2})$	M1
			but $\cos^2(m(2)) = (1 - \sqrt{2})^2 = 3 - 2\sqrt{2}$	M1
			which is greater, as $(3 - 2\sqrt{2}) - \frac{1}{4}(2 - \sqrt{2}) = \frac{1}{4}(10 - 7\sqrt{2})$	M1
			$= \frac{1}{2(10 + 7\sqrt{2})} > 0$	A1
			but both values are between $\frac{1}{2}\pi$ and π,	M1
			so both cosines are negative and hence $\cos^2(m(2)) > \cos^2(\mu(2)) \Rightarrow$ $0 > \cos(\mu(2)) > \cos(m(2))$	M1
			so $m(2) > \mu(2)$	A1
				[10]