

STEP II, 2023, Q12 MS

Question	Answer	Mark
12 (i)	$P(Y \leq t) = P(X_i \leq t \text{ for all } i = 1, \dots, n)$	E1
	$= \prod_{i=1}^n P(X_i \leq t) = [P(X_1 \leq t)]^n$	M1
	$= \left(\int_0^t \frac{1}{2} \sin x \, dx \right)^n = \frac{1}{2^n} (1 - \cos t)^n$	A1
	so $f_Y(t) = \frac{n \sin t}{2^n} (1 - \cos t)^{n-1}$	[3]
(ii)	$\frac{1}{2^n} (1 - \cos m(n))^n = \frac{1}{2}$	M1
	so $m(n) = \arccos \left(1 - 2^{\frac{n-1}{n}} \right)$	A1
	which tends to π as $n \rightarrow \infty$.	A1
		[3]
(iii)	$\mu(n) = \int_0^\pi x \frac{n}{2^n} \sin x (1 - \cos x)^{n-1} \, dx$	M1
	$= \left[x \frac{1}{2^n} (1 - \cos x)^n \right]_0^\pi - \int_0^\pi \frac{1}{2^n} (1 - \cos x)^n \, dx$	
	$= \left(\pi \frac{1}{2^n} 2^n - 0 \right) - \int_0^\pi \frac{1}{2^n} (1 - \cos x)^n \, dx$ as required	A1
		[2]
(a)	As $\mu(n) = \pi - \int_0^\pi \left(\frac{1 - \cos x}{2} \right)^n \, dx$, the integrand decreases with n throughout $(0, \pi)$	M1
	and so $\mu(n)$ increases with n	A1
		[2]
(b)	$\mu(2) = \pi - \int_0^\pi \frac{1}{4} (1 - 2 \cos x + \cos^2 x) \, dx$	M1
	$= \pi - \int_0^\pi \frac{1}{4} (1 - 2 \cos x + \frac{1}{2}(1 + \cos 2x)) \, dx$	
	$= \pi - \left[\frac{3}{8}x - \frac{1}{2} \sin x + \frac{1}{16} \sin 2x \right]_0^\pi$	M1
	$= \frac{5}{8}\pi$	A1
	so $\cos^2(\mu(2)) = \frac{1}{2} \left(1 + \cos \frac{5}{4}\pi \right) = \frac{1}{4} (2 - \sqrt{2})$	M1
	but $\cos^2(m(2)) = (1 - \sqrt{2})^2 = 3 - 2\sqrt{2}$	M1
	which is greater, as $(3 - 2\sqrt{2}) - \frac{1}{4}(2 - \sqrt{2}) = \frac{1}{4}(10 - 7\sqrt{2})$	M1
	$= \frac{1}{2(10 + 7\sqrt{2})} > 0$	A1
	but both values are between $\frac{1}{2}\pi$ and π ,	M1
	so both cosines are negative and hence $\cos^2(m(2)) > \cos^2(\mu(2)) \Rightarrow 0 > \cos(\mu(2)) > \cos(m(2))$	M1
	so $m(2) > \mu(2)$	A1
		[10]



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