STEP II, 2023, Q10 MS

Question			Answer	Mark
10	(i)		$z = ut\sin\alpha - \frac{1}{2}gt^2$ and $x = ut\cos\alpha$	M1
			require $ut\sin\alpha - \frac{1}{2}gt^2 = A - B(ut\cos\alpha)^2$ to have a double root	M1
			so $\left(Bu^2\cos^2\alpha - \frac{1}{2}g\right)t^2 + (u\sin\alpha)t - A = 0$ has zero discriminant	A1
			so $u^2\sin^2\alpha + 4A\left(Bu^2\cos^2\alpha - \frac{1}{2}g\right) = 0$	A1
			using $\sin^2\alpha = 1 - \cos^2\alpha$, gives $u^2 - u^2\cos^2\alpha + 4ABu^2\cos^2\alpha - 2Ag = 0$	A1
			If this is true for all α, both the left-hand side, and the coefficient of $\cos^2\alpha$ on the right-hand side must be zero	M1
			So $A = \frac{u^2}{2g}$ and $B = \frac{g}{2u^2}$	A1
				[7]
	(ii)		The set of points inside and on S are vulnerable	M1
			because if a point outside S was on a trajectory, it would have to cross S, which contradicts the fact that it touches to the surface.	A1
				[2]
	(iii)		Particles in the *x*-*z* plane have, eliminating α,	M1
			$$\left(z + \frac{1}{2}gt^2\right)^2 + x^2 = u^2t^2$$	A1
			so, rotating this circle about the *z*-axis, all the particles lie on a sphere	M1
			of radius *ut* and centre $\left(0,\ 0,\ -\frac{1}{2}gt^2\right)$	A1
				[4]
	(iv)		for Q, $z = \frac{u^2}{2g} - \frac{1}{2}gt^2$ and $x = ut$	M1
			so $z = \frac{u^2}{2g} - \frac{gx^2}{2u^2}$, which is the equation of *E*	A1
				[2]
	(v)		Using **(i)**, P_α meets *E* at time *t* given by	M1
			$$g^2\sin^2\alpha\, t^2 - 2gu\sin\alpha\, t + u^2 = 0$$	A1
			so at $t = \frac{u}{g\sin\alpha}$	A1
			so require $u(t - T) = u\cos\alpha\, t$	M1
			that is $T = \frac{u}{g}\frac{1-\cos\alpha}{\sin\alpha}$	A1
				[5]