STEP II, 2022, Q9

9 A rectangular prism is fixed on a horizontal surface. A vertical wall, parallel to a vertical face of the prism, stands at a distance d from it. A light plank, making an acute angle θ with the horizontal, rests on an upper edge of the prism and is in contact with the wall below the level of that edge of the prism and above the level of the horizontal plane. You may assume that the plank is long enough and the prism high enough to make this possible.

The contact between the plank and the prism is smooth, and the coefficient of friction at the contact between the plank and the wall is μ. When a heavy point mass is fixed to the plank at a distance x, along the plank, from its point of contact with the wall, the system is in equilibrium.

(i) Show that, if $x = d\sec^3\theta$, then there is no frictional force acting between the plank and the wall.

(ii) Show that, if $x > d\sec^3\theta$, it is necessary that

$$\mu \geqslant \frac{x - d\sec^3\theta}{x\tan\theta}$$

and give the corresponding inequality if $x < d\sec^3\theta$.

(iii) Show that

$$\frac{x}{d} \geqslant \frac{\sec^3\theta}{1+\mu\tan\theta}.$$

Show also that, if $\mu < \cot\theta$, then

$$\frac{x}{d} \leqslant \frac{\sec^3\theta}{1-\mu\tan\theta}.$$

(iv) Show that if x is such that the point mass is fixed to the plank somewhere between the edge of the prism and the wall, then $\tan\theta < \mu$.