STEP II, 2022, Q7

7 Let $h(z) = nz^6 + z^5 + z + n$, where z is a complex number and $n \geqslant 2$ is an integer.

(i) Let w be a root of the equation $h(z) = 0$.

(a) Show that $|w^5| = \sqrt{\dfrac{f(w)}{g(w)}}$, where

$$f(z) = n^2 + 2n\mathrm{Re}(z) + |z|^2 \text{ and } g(z) = n^2|z|^2 + 2n\mathrm{Re}(z) + 1.$$

(b) By considering $f(w) - g(w)$, prove by contradiction that $|w| \geqslant 1$.

(c) Show that $|w| = 1$.

(ii) It is given that the equation $h(z) = 0$ has six distinct roots, none of which is purely real.

(a) Show that $h(z)$ can be written in the form

$$h(z) = n(z^2 - a_1 z + 1)(z^2 - a_2 z + 1)(z^2 - a_3 z + 1),$$

where a_1, a_2 and a_3 are real constants.

(b) Find $a_1 + a_2 + a_3$ in terms of n.

(c) By considering the coefficient of z^3 in $h(z)$, find $a_1 a_2 a_3$ in terms of n.

(d) How many of the six roots of the equation $h(z) = 0$ have a negative real part? Justify your answer.