STEP II, 2022, Q6 MS

Question			Answer	Mark
6	(i)	(a)	Differentiating implicitly with respect to x gives $2x + 2y\frac{dy}{dx} = 2a$ so, by substitution, $x^2 + y^2 = x\left(2x + 2y\frac{dy}{dx}\right)$	B1
			For second family: $2x + 2y\frac{dy}{dx} = 2b\frac{dy}{dx}$	M1
			so $y\left(2x + 2y\frac{dy}{dx}\right) = (x^2 + y^2)\frac{dy}{dx}$	A1
			$(x^2 - y^2)\frac{dy}{dx} = 2xy$	A1
				[4]
		(b)	The product of the gradients at points (x, y) where the curves meet is $\frac{y^2-x^2}{2xy} \times \frac{2xy}{x^2-y^2} = -1$, provided $x \neq y$, So the tangents to the curves at these points are perpendicular	B1
			At (c, c), for the first family of curves: $$2c^2\frac{dy}{dx} = 0$$ and so $\frac{dy}{dx} = 0$. For the second family of curves: $$2c^2\frac{dx}{dy} = 0$$ and so $\frac{dy}{dx} = \infty$.	M1
			and the tangents to the circles $(x - c)^2 + y^2 = c^2$ and $(y - c)^2 + x^2 = c^2$ at this point are $y = c$ and $x = c$.	A1
			which are indeed perpendicular.	A1
				[4]
	(ii)		First family $\frac{dy}{dx} = \frac{c}{x}$	M1
			so $x \ln x \frac{dy}{dx} = y$	A1
			so orthogonal family has $y\frac{dy}{dx} = -x \ln x$	A1
			solving differential equation by separating variables	M1
			$\int -x \ln x \, dx = -\frac{1}{2}x^2 \ln x - \int -\frac{1}{2}x^2 \cdot \frac{1}{x} dx$	M1
			$= -\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2$	A1
			so $\frac{1}{2}y^2 = -\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + c$	A1
				[7]
	(iii)		If two curves, with parameters k_1, k_2 meet, require $4k_1(x + k_1) = 4k_2(x + k_2)$	M1
			so $x = -(k_1 + k_2)$	A1
			$y^2 = -4k_1k_2$	A1
			for any curve, $2y\frac{dy}{dx} = 4k$	M1
			so the gradients of the two curves satisfy $\left.\frac{dy}{dx}\right\|_1 \cdot \left.\frac{dy}{dx}\right\|_2 = \frac{2k_1}{y}\frac{2k_2}{y} = -1$	A1 CSO
				[5]