STEP II, 2021, Q6 MS

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(i)	The shortest distance from O to the line AB is $(R+w)\cos\alpha$	B1
	Since $\frac{1}{3}\pi \le \alpha \le \frac{1}{2}\pi$, $0 \le \cos\alpha \le \frac{1}{2}$.	M1
	Since $w < R$, $(R+w)\cos\alpha < \frac{1}{2}(R+R) = R$, so the midpoint of the line AB lies inside the smaller circle.	E1
(ii) (a)	$$(R+d)^2 = (R+w)^2 + d^2 - 2d(R+w)\cos(\pi-\alpha)$$	M1 A1
	$$R^2 + 2Rd + d^2 = R^2 + 2Rw + w^2 + d^2 + 2d(R+w)\cos\alpha$$ $$d = \frac{w(2R+w)}{2(R-(R+w)\cos\alpha)}$$	M1 A1
(b)	$$\angle O'AO = \alpha - \theta$$ $$\frac{\sin(\alpha-\theta)}{d} = \frac{\sin(\pi-\alpha)}{R+d}$$	M1
	$$\sin(\alpha-\theta) = \frac{d\sin\alpha}{R+d}$$	A1
(iii)	$$\frac{d}{R} = \frac{\left(\frac{w}{R}\right)\left(2+\frac{w}{R}\right)}{2\left(1-\left(1+\frac{w}{R}\right)\cos\alpha\right)} \approx \frac{1}{1-\cos\alpha} \times \frac{w}{R}$$	M1 A1
	$1-\cos\alpha > \frac{1}{2}$ and $\frac{w}{R}$ is much less than 1, so $\frac{d}{R}$ is much less than 1.	E1
	$$\sin(\alpha-\theta) = \frac{\left(\frac{d}{R}\right)\sin\alpha}{1+\left(\frac{d}{R}\right)} < \frac{d}{R}$$	M1
	$\sin(\alpha-\theta)$ is much less than 1 and so $(\alpha-\theta)$ is a small angle.	M1
	Therefore $\sin(\alpha-\theta) \approx \alpha-\theta$, so $\alpha-\theta$ is much less than 1.	E1
(iv)	The longer length is $(R+w) \times 2\alpha$ The shorter length is $(R+d) \times 2\theta$	
	$$S = 2\alpha(R+w) - 2\theta(R+d)$$ $$S = 2(R+d+w-d)\alpha - 2(R+d)\theta$$ $$S = 2(R+d)(\alpha-\theta) + 2(w-d)\alpha$$	B1
	$$\alpha-\theta \approx \frac{w\sin\alpha}{R(1-\cos\alpha)}$$	M1
	$$d-w \approx \frac{\cos\alpha}{(1-\cos\alpha)} \times \frac{w}{R}$$	M1
	So $S \approx 2(R+d)\frac{w\sin\alpha}{R(1-\cos\alpha)} - 2\left(\frac{\cos\alpha}{(1-\cos\alpha)} \times \frac{w}{R}\right)\alpha$ As a fraction of the longer path length:	
	$$\frac{S}{2\alpha(R+w)} = \frac{R+d}{R+w} \times \frac{\alpha-\theta}{\alpha} + \frac{w-d}{R+w} \approx \frac{\sin\alpha}{\alpha(1-\cos\alpha)}\frac{w}{R} - \frac{\cos\alpha}{(1-\cos\alpha)}\frac{w}{R}$$	M1
	$$S \approx \left(\frac{\sin\alpha - \alpha\cos\alpha}{\alpha(1-\cos\alpha)}\right)\frac{w}{R} \quad \textbf{AG}$$	A1

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