STEP II, 2021, Q5 MS

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(i)

$$\frac{dy}{dx} = (x-a)\frac{du}{dx} + u$$ **M1** **A1**

$$(x-a)\left((x-a)\frac{du}{dx} + u\right) = (x-a)u - x$$

$$(x-a)^2\frac{du}{dx} = -x$$

$$u = \int \frac{-x}{(x-a)^2}dx = \int \frac{-(x-a)-a}{(x-a)^2}dx$$ **M1**

$$u = -\ln|x-a| + \frac{a}{x-a} + c$$ **A1**

$$y = -(x-a)\ln|x-a| + a + c(x-a)$$ **A1 (ft)**

(ii)

(a) The gradient of the line through $(1,t)$ and $(t, f(t))$ is $\frac{f(t)-t}{t-1} = f'(t)$ **M1**

Applying the result from (i), with a=1 or solving the d.e. directly:

$$f(x) = -(x-1)\ln|x-1| + 1 + c(x-1)$$ **B1 (ft)**

$f(0) = 0$, so $c = 1$ **B1 (ft)**

$$y = -(x-1)\ln|x-1| + x$$

$$\frac{dy}{dx} = -\ln|x-1|$$ **M1**

$-\ln|x-1| = 0$ when $x = 0$ only (since $x < 1$) and $y = 0$ **A1 (ft)**

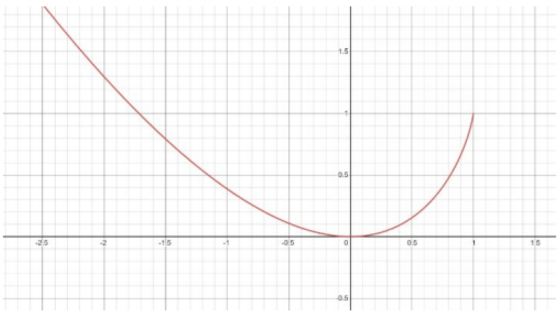
As $x \to 1^-$, $y \to 1^-$ and $\frac{dy}{dx} \to \infty$. **B1 (ft)**

Sketch showing:

Curve approaching $(1,1)$ with a vertical tangent at that point. **G1 (ft)**

Minimum point at $(0,0)$. **G1 (ft)**

$y \to \infty$ as $x \to -\infty$ **G1 (ft)**



(b) $f(2) = 2$, so $c = 1$ **B1 (ft)**

$$y = -(x-1)\ln|x-1| + x$$
$$\frac{dy}{dx} = -\ln|x-1|$$

$-\ln|x-1| = 0$ when $x = 2$ only (since $x > 1$) and $y = 2$. **B1 (ft)**

As $x \to 1^+$, $y \to 1^+$ and $\frac{dy}{dx} \to \infty$. **B1 (ft)**

Sketch showing:

Curve approaching $(1,1)$ with a vertical tangent at that point. **G1 (ft)**

Maximum point at $(2,2)$. **G1 (ft)**

The curve crossing the x-axis for some $x > 2$ and $y \to -\infty$ as $x \to \infty$ **G1 (ft)**