STEP II, 2021, Q12 MS

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(i) Player A wins the match on game n with probability $p_A(1 - p_A - p_B)^{n-1}$ **B1**

The probability that A wins the match is the sum to infinity of a geometric series with **M1**

$a = p_A$, $r = 1 - p_A - p_B$ **M1**

$$\frac{p_A}{p_A + p_B} \quad \textbf{AG}$$ **M1** **A1**

(ii) The difference between the number of games won by the two players is initially 0 and either increases or decreases by 1 after each game. **E1**

Therefore, it can only be an even number (and so the match can only be won) after an even number of games. **E1**

Considering pairs of turns at a time **M1**

The game is equivalent to that in part (i), with $p_A = p^2$ and $p_B = q^2$, **M1**

and $0 < p_A + p_B < 1$ **M1**

so the probability that A wins the match is

$$\frac{p^2}{p^2 + q^2} \quad \textbf{AG}$$ **A1**

(iii) Version 1:

The player has to win round 1 for the game to continue (with probability p). **M1**

Following that the game is equivalent to that in part (ii), so the probability that the player wins overall is **M1**

$$\frac{p^3}{p^2 + q^2}$$ **A1**

Version 2:

The only way for the player to win is by winning two rounds in a row, so with probability **M1**

$$p^2$$ **A1**

$$p^2 - \frac{p^3}{p^2 + q^2} = \frac{p^4 + p^2q^2 - p^3}{p^2 + q^2}$$ **M1**

$$= \frac{p^4 + p^2 - 2p^3 + p^4 - p^3}{p^2 + q^2}$$

$$= \frac{2p^4 - 3p^3 + p^2}{p^2 + q^2}$$

$$\frac{p^2(2p - 1)(p - 1)}{p^2 + q^2}$$ **M1**

If $1 > p > \frac{1}{2}$, $\frac{p^2(2p-1)(p-1)}{p^2+q^2} < 0$, so the player is more likely to win in version 1 (the cautious version) **AG** **E1**

If $0 < p < \frac{1}{2}$, $\frac{p^2(2p-1)(p-1)}{p^2+q^2} > 0$, so the player is more likely to win in version 2 (the bold version) **AG** **E1**